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Converse and Smoothness Theorems for Erdős Weights in L_p $(0<p\leqslant\infty)$

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We prove converse and smoothness theorems of polynomial approximation in weighted L_p spaces with norm $\|fW\|_{L_p(\mathbb{R})}$ $(0<p\leqslant\infty)$ for Erdős weights on the real line. In particular we prove characterization theorems involving realization functionals and thereby establish some interesting properties of our weighted modulus of continuity.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $W:=\exp(-Q)$ where Q: $\mathbb{R}\to\mathbb{R}$ is even and is of faster than polynomial growth at infinity. Then W is called an Erdős weight.

Archetypal examples of such weights are

(a) $$W_{k,\alpha}(x):=\exp(-\exp_k(|x|^\alpha)),\qquad \alpha>1,\quad k\geqslant 1, \tag{1.1}$$

where $\exp_k(\ )=\exp(\exp(\cdots(\exp(\ ))))$ denotes the kth iterated exponential.

(b) $$W_{A,\beta}(x):=\exp(-\exp(\log(A+x^2)^\beta)), \tag{1.2}$$

where $\beta>1$ and A is large enough.

For more on the subject, we refer the reader to [16, 18] and the references cited therein.

Recently, we investigated Jackson theorems for large classes of Erdős weights in L_p $(0<p\leqslant\infty)$ [2]. More precisely, we estimated how fast

$$E_n[f]_{W,p}:=\inf_{P\in\mathscr{P}_n}\|(f-P)W\|_{L_p(\mathbb{R})}\to 0,\qquad n\to\infty.$$

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Here $E_n[f]_{W,p}$ is the error of best weighted approximation for suitable $f: \mathbb{R} \to \mathbb{R}$ and $\mathscr{P}_n$ denotes the class of polynomials of degree at most n.

Direct and converse theorems for rates of approximation are an extensively researched and widely studied subject. For weights on $\mathbb{R}$, analogues of Jackson–Bernstein theorems were initiated by Dzrbasjan, but were more intensively studied by Freud in the 1960s–1970s [10, 11, 23]. Since then, their ideas have been generalized and extended by many. See [2, 7–9, 12, 19] and the references cited therein.

In this paper, we investigate converse theorems of polynomial approximation for Erdős weights. To state our results, we need a suitable class of weights and various quantities.

Throughout, $C, C_1, C_2, \ldots$, will denote positive constants independent of n, x and $P \in \mathscr{P}_n$ not necessarily the same in different occurrences. We write $C \neq C(L)$ to mean that the constant is independent of L.

Moreover for real sequences A_n and $B_n \not\equiv 0$, $A_n = O(B_n)$, $A_n \sim B_n$, and $A_n = o(B_n)$ will mean respectively that there exist constants $C_1, C_2, C_3 > 0$ independent of n such that $A_n/B_n \leqslant C_1$, $C_2 \leqslant A_n/B_n \leqslant C_3$ and $\lim_{n\to\infty} |A_n/B_n| = 0$. Similar notation will be used for functions and sequences of functions.

We shall say that a function

$$f: (a, b) \to (0, \infty)$$

is quasi-increasing if $\exists C > 0$ such that

$$a < x < y < b \Rightarrow f(x) \leqslant Cf(y).$$

We need a suitable class of weights:

Definition 1.1. Let $W(x) := \exp[-Q(x)]$ where $Q: \mathbb{R} \to \mathbb{R}$ is even and continuous satisfying,

(a) $xQ'(x)$ is strictly increasing in $(0, \infty)$ with

$$\lim_{x\to 0^+} xQ'(x) = 0.$$

(b) The function

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{1.3}$$

is quasi-increasing in (C, ∞) for some $C > 0$ and

$$\lim_{x\to\infty} T(x) = \infty. \tag{1.4}$$

(c) Assume

$$\frac{yQ'(y)}{xQ'(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_3}, \qquad y \geqslant x \geqslant C_2. \tag{1.5}$$

for some C_1, C_2, $C_3 > 0$. Then Q is called the external field associated with W and we write $W \in \mathscr{E}_1$.

Some Remarks. (a) The function T serves as a measure of the regularity of growth of Q. In particular, it is not difficult to show that (1.4) forces Q to be of faster than polynomial growth at infinity.

(b) We need the condition that $xQ'(x)$ be strictly increasing in order to ensure the existence of the *Mhaskar–Rakhmanov–Saff* number, a_u defined as the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)\, dt}{\sqrt{1-t^2}}, \qquad u > 0. \tag{1.6}$$

For those unfamiliar, the quantity (PW), $P \in \mathscr{P}_n$ "lives" most of the time in $[-a_n, a_n]$. We refer the interested reader to [17, 21, 26] for more on a_n and its "cousin" q_n, the Freud number. For a different perspective on discrete sets and to concave external fields, we refer the reader to [4,5]. For Erdős weights, a_n has the effect that although $Q(x)$ might grow very rapidly for large x, $Q(a_u)$ does not exceed a positive power of u. For example, for $W_{k,\alpha}$, a_u grows like $(\log_k u)^{1/\alpha}$ where $\log_k(\ ) = \log(\log(\cdots(\log(\ ))))$ denotes the kth iterated logarithm.

(c) Inequality (1.5) is a weak regularity condition on T, for one has typically for each $\varepsilon > 0$,

$$T(x) = O(\log Q'(x))^{1+\varepsilon}, \qquad x \to \infty. \tag{1.7}$$

For example, for $W_{k,\alpha}(x)$,

$$T(x) = \alpha x^\alpha \left[\prod_{j=1}^{k-1} \exp_j(x^\alpha)\right],$$

so that C_3 can be made arbitrarily close to 1. This is also the case for $W_{A,\beta}$.

We proceed to define our modulus of continuity and realization functional as in [1–3].

For $h>0$, an interval J, $r\geqslant 1$, and $f\colon \mathbb{R}\to\mathbb{R}$ we define

$$\Delta_h^r(f, x, J) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x+\frac{rh}{2}-ih\right), & x\pm\frac{rh}{2}\in J \\ 0, & \text{otherwise} \end{cases} \tag{1.8}$$

to be the rth symmetric difference of f. If J is not specified, it can be taken as $\mathbb{R}$.

Following ideas of [9], to reflect endpoint effects in our approximation, we need our increment h in (1.8) to depend on x and in particular on the function,

$$\Phi_t(x) := \left|1-\frac{|x|}{\sigma(t)}\right|^{1/2} + T(\sigma(t))^{-1/2}, \qquad x\in\mathbb{R}, \tag{1.9}$$

where

$$\sigma(t) := \inf\left\{a_u\colon \frac{a_u}{u}\leqslant t\right\} \tag{1.10}$$

and $t>0$ but is typically small enough.

An easy way to understand σ is to see it as the inverse of the map

$$u\colon \to \frac{a_u}{u}$$

which decays to zero as $u\to\infty$. Clearly σ is decreasing.

We may then define our weighted modulus of continuity for $0<p\leqslant\infty$ and $r\geqslant 1$ by

$$\begin{aligned} w_{r,p}(f, W, t) := {} & \sup_{0<h\leqslant t} \|W(\Delta^r_{h\Phi_t(x)}(f))\|_{L_p(|x|\leqslant\sigma(2t))} \\ & + \inf_{R \text{ of deg} \leqslant r-1} \|(f-R)\,W\|_{L_p(|x|\geqslant\sigma(4t))}. \end{aligned} \tag{1.11}$$

Further, we define its averaged "cousin,"

$$\begin{aligned} \bar{w}_{r,p}(f, W, t) := {} & \left(\frac{1}{t}\int_0^t \|W(\Delta^r_{h\Phi_t(x)}(f))\|^p_{L_p(|x|\leqslant\sigma(2t))}\,dh\right)^{1/p} \\ & + \inf_{R \text{ of deg} \leqslant r-1} \|(f-R)\,W\|_{L_p(|x|\geqslant\sigma(4t))} \end{aligned} \tag{1.12}$$

(if $p=\infty$ we set $\bar{w}_{r,p}=w_{r,p}$).

Clearly $\bar{w}_{r,p}(f, W, t)\leqslant w_{r,p}(f, W, t)$.

Some Remarks Concerning Our Modulus. (a) Although at first difficult to assimilate, we see that the definition of σ in (1.10) is natural, as at least for purposes of approximation by polynomials of degree $\leqslant n$, we may think of $t = a_n/n$ (recall t is small) so that $\sigma(t)$ grows like a_n. Following [9], our modulus consists of two parts. The "main" part involves rth symmetric differences over the interval $[-a_{n/2}, a_{n/2}]$. The "tail" involves an error of weighted polynomial approximation over the remainder of $\mathbb{R}$ and is necessary because of the inability of $(P_n W)$ to approximate beyond $[-a_{n/2}, a_{n/2}]$. Its presence ensures that at least for $f \in \mathcal{P}_{r-1}$,

$$w_{r,p}(f, W, t) \equiv 0.$$

For converse saturation type results, we refer the reader to [3].

(b) We note that the function Φ_t describes the improvement in the degree of approximation near $\pm a_{n/2}$, in much the same way that $\sqrt{1-x^2}$ does for weights on $[-1, 1]$.

(c) We observe that unlike the moduli in [8, 9], our modulus w is not necessary monotone increasing in t. This created severe difficulties in our analysis. The results of [2] show that under additional assumptions on W it is possible to replace our modulus by one that is increasing in t; however, for $\mathcal{E}_1$ this is an open question.

In [2], we proved the following Jackson theorems:

THEOREM 1.2. *Let $W \in \mathcal{E}_1$, $r \geqslant 1$, and $0 < p \leqslant \infty$. Then for all $f: \mathbb{R} \to \mathbb{R}$ for which $fW \in L_p(\mathbb{R})$ (and for $p = \infty$, we require f to be continuous, and fW to vanish at $\pm\infty$), we have for $n \geqslant C$,*

$$E_n[f]_{W,p} \leqslant C_1 \bar{w}_{r,p}\left(f, W, C_2 \frac{a_n}{n}\right) \leqslant C_1 w_{r,p}\left(f, W, C_2 \frac{a_n}{n}\right), \tag{1.13}$$

where the C_j, $j = 1, 2$, are independent of f and n.
Moreover, given $\lambda(n) \in [\frac{4}{5}, 1]$,

$$E_n[f]_{W,p} \leqslant C_1 \bar{w}_{r,p}\left(f, W, C_2 \lambda(n) \frac{a_n}{n}\right) \leqslant C_1 w_{r,p}\left(f, W, C_2 \lambda(n) \frac{a_n}{n}\right). \tag{1.14}$$

Some remarks. (a) The result above indicated a *Nikolskii–Timan–Brudnyi* effect whereby, as in weights on $[-1, 1]$, we have better approximation towards the endpoints of the Mhaskar–Rakhmanov–Saff interval.

(b) We remark that with a little extra effort, we may replace C in (1.13) by $r-1$ (cf. [3]).

In establishing our converse theorems, we need the notion of the K-functional. While K-functionals were introduced in the context of interpolation of spaces, one of their most important applications has been in the analysis of moduli of continuity, and in converse theorems in approximation theory. J. Peetre first made the connection between his K-functional and the modulus of continuity in 1968. His ideas have been generalized and extended by many including Ditzian, Freud, Hristov, Ivanov, Lubinsky, Mhaskar, and Totik. We refer the reader to [8–12] and the references cited therein.

The Ditzian–Totik rth order K-Functional has the form

$$K^{\wedge}_{r,p}(f, W, t^r) := \inf_{\substack{g \\ g^{(r-1)} \text{ locally absolutely} \\ \text{continuous}}} \{\|(f-g)W\|_{L_p(\mathbb{R})} + t^r \|g^{(r)}W\|_{L_p(\mathbb{R})}\}. \tag{1.15}$$

Here, $t>0$, $r \geqslant 1$, and $p \geqslant 1$.

We may think of the second term of (1.15) measuring the smooth part of f and the first part measuring the distance of f to that smooth part [9]. The idea, following a general technique of Ditzian, Hristov, and Ivanov [7], is to prove inequalities of the form

$$w^{\wedge}_{r,p}(f, W, \alpha t) \leqslant C_1 K^{\wedge}_{r,p}(f, W, t^r) \leqslant C_2 w^{\wedge}_{r,p}(f, W, t) \tag{1.16}$$

for a suitable modulus $w^{\wedge}_{r,p}(f, .)$. Here $\alpha>0$ is fixed in advance, $C_1, C_2>0$, and t is small enough.

Unfortunately, $K^{\wedge} \equiv 0$ in L_p $(0<p<1)$ [7], so we need the notion of a realization functional, a concept attributed to Hristov and Ivanov. Our realization functional has the form

$$K_{r,p}(f, W, t^r) := \inf_{P \in \mathcal{P}_n} \{\|(f-P)W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)}\Phi^r_t W\|_{L_p(\mathbb{R})}\}, \tag{1.17}$$

where $t>0$, $0<p \leqslant \infty$, and $r \geqslant 1$ are chosen in advance and

$$n = n(t) := \inf\left\{k: \frac{a_k}{k} \leqslant t\right\}. \tag{1.18}$$

Further define the ordinary K-functional by

$$K^{*}_{r,p}(f, W, t^r) := \inf_{\substack{g \\ g^{(r-1)} \text{ locally absolutely} \\ \text{continuous}}} \{\|(f-g)W\|_{L_p(\mathbb{R})} + t^r \|g^{(r)}\Phi^r_t W\|_{L_p(\mathbb{R})}\}. \tag{1.19}$$

We begin with our main equivalence result:

THEOREM 1.3. *Let $W \in \mathcal{E}_1$, $L, \alpha > 0$, $r \geqslant 1$, $0 < p \leqslant \infty$, and f as in Theorem* 1.2. *Assume that there is a Markov–Bernstein inequality of the form*

$$\| R_n' \Phi_{a_n/n} W \|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \| R_n W \|_{L_p(\mathbb{R})}, \qquad 0 < p \leqslant \infty, \quad R_n \in \mathcal{P}_n, \tag{1.20}$$

where $C \neq C(n, R_n)$. Then $\exists C_1, C_2, C_3 > 0$ independent of f and t such that for $t \in (0, t_0)$,

$$\text{(a)} \quad w_{r,p}(f, W, Lt) \leqslant C_1 K_{r,p}(f, W, t^r) \leqslant C_2 w_{r,p}(f, W, C_3 t). \tag{1.21}$$

Moreover, uniformly for t and f,

$$\text{(b)} \quad w_{r,p}(f, W, t) \sim \bar{w}_{r,p}(f, W, t) \sim K_{r,p}(f, W, t^r) \tag{1.22}$$

and

$$\text{(c)} \quad w_{r,p}(f, W, \alpha t) \sim w_{r,p}(f, W, t). \tag{1.23}$$

Note that the constant in the $\sim$ relation in (1.23) depends on α. For the exact dependence, we refer the interested reader to [3].

Remark. (a) The Markov inequality (1.20) is true for $W \in \mathcal{E}_1$ [15]. For this reason, we dispense with the proof here and assume the result. We refer the interested reader to [8, 19] where similar assumptions were made.

(b) Inequality (1.20) was proved for $p = \infty$ in [18] and for $0 < p < \infty$ in [20] under additional conditions on Q, namely conditions on Q'' which are satisfied for $W_{k,\alpha}$ and $W_{A,\beta}$ given by (1.1) and (1.2).

(c) We finally note that for $p \geqslant 1$, the methods of [9] should enable one to avoid assuming (1.20) altogether. However, as it is needed in the later corollaries, we do not pursue this idea further here.

Theorem 1.3 allows us to deduce a simpler Jackson theorem to Theorem 1.2:

COROLLARY 1.4. *Assume the hypotheses of Theorem* 1.3. *Then we have for $n \geqslant C_1$,*

$$E_n[f]_{W,p} \leqslant C_2 \bar{w}_{r,p}\left(f, W, \frac{a_n}{n}\right) \leqslant C_2 w_{r,p}\left(f, W, \frac{a_n}{n}\right). \tag{1.24}$$

Here, C_2 is independent of f and n.

We note that the point of this corollary is that we have removed the constant from inside the modulus in (1.13) and (1.14).

We have the following converse theorems:

THEOREM 1.5. *Assume the hypotheses of Theorem* 1.3. *Let* $q = \min\{1, p\}$. *For* $0 < t < C$, *determine* $n = n(t)$ *by* (1.18) *and let* $l = [\log_2 n] =$ *the largest integer* $\leqslant \log_2 n$. *Then we have*

$$w_{r,p}(f, W, t) \leqslant C_1 t^r \left[\sum_{k=-1}^{l} (l-k+1)^{rq/2} \left(\frac{2^k}{a_{2^k}} \right)^{rq} E_{2^k}[f]_{W,p}^q \right]^{1/q}, \tag{1.25}$$

where $C_1 \neq C_1(f, t)$ *and where we set* $E_{2^{-1}} = E_{2^0}$.

We deduce

COROLLARY 1.6. *Assume the hypotheses of Theorem* 1.3. *Then for every* $0 < \alpha < r$ *the following are equivalent*:

$$\text{(a)} \quad w_{r,p}(f, W, t) = O(t^\alpha), \qquad t \to 0^+, \tag{1.26}$$

$$\text{(b)} \quad K_{r,p}(f, W, t^r) = O(t^\alpha), \qquad t \to 0^+,$$

$$\text{(c)} \quad E_n[f]_{W,p} = O\left(\frac{a_n}{n}\right)^\alpha, \qquad n \to \infty. \tag{1.27}$$

Remark. We remark that a different characterization appears in [3] where α is allowed to equal r.

Finally, we obtain estimates of our modulus in terms of $f^{(r)}$ and deduce the equivalence of the K-functional with the realization functional for $p \geqslant 1$.

COROLLARY 1.7. *Let* $1 \leqslant p \leqslant \infty$ *and assume the hypotheses of Theorem* 1.3.

(a) *If* $f^{(r)} W \in L_p(\mathbb{R})$, *we have for* $t \in (0, C_2)$,

$$w_{r,p}(f, W, t) \leqslant C_1 t^r \, \|f^{(r)} \Phi_t^r W\|_{L_p(\mathbb{R})}. \tag{1.28}$$

Here $C_j \neq C_j(f, t)$, $j = 1, 2$.

(b) *We have for* $t \in (0, C_3)$,

$$1 \leqslant K^*_{r,p}(f, W, t)/K_{r,p}(f, W, t) \leqslant C_4. \tag{1.29}$$

Here $C_j \neq C_j(f, t)$, $j = 3, 4$.

Remark. We remark that (1.28) is false for $0 < p < 1$. Indeed set for $\varepsilon \in (0, \frac{1}{2})$

$$f_\varepsilon(x) := \begin{cases} 0, & x \in [-1, 0] \\ \varepsilon^{-1} x, & x \in (0, \varepsilon] \\ 1, & x \in (\varepsilon, 1]. \end{cases}$$

Then $fW \in L_p$ $(0<p<1)$, f is of compact support and so it is easy to see that for fixed $t>0$, there exists $C=C(t, W)>0$ such that

$$w_{r,p}(f_\varepsilon, W, t) > C$$

and

$$\|f'_\varepsilon \Phi_t W\|_{L_p(\mathbb{R})} \to 0, \qquad \varepsilon \to 0^+.$$

An Important Note on the Structure of This Paper. Sections 2 and 3, establish some machinery required for the entire paper. This includes, in particular, an extension of the Markov–Bernstein inequality (1.20). Many of the proofs are technical and serve merely as tools for the proofs of our main results. Thus, we suggest the reader skip these sections at first and return to them at the end of the paper. In Section 4, we prove a theorem required for the lower bound in Theorem 1.3, whereby we approximate polynomials of degree n, $n \geqslant 1$ by those of degree $r-1$, $r \geqslant 1$. This technique, although similar to that used in [8], is new for Erdős weights on $\mathbb{R}$ and $[-1, 1]$ and we believe it to be of independent interest. In Section 5 we prove Theorem 1.3 and Corollary 1.4 and in Section 6 we prove Theorem 1.5 and Corollaries 1.6 and 1.7.

2. TECHNICAL LEMMAS

LEMMA 2.1. *Let $W \in \mathscr{E}_1$. Then*

(a) *Given $A \geqslant 0$, the functions $Q'(u)\,u^{-A}$ and $Q(u)\,u^{-A}$ are quasi-increasing and increasing respectively for large enough u.*

(b) *a_u is uniquely defined for $u \in (0, \infty)$. Furthermore, it is a strictly increasing function of u.*

(c) *We have for u large enough and $\alpha > 0$*

$$\begin{aligned} &\text{(i)} \quad a_u Q'(a_u) \sim u T(a_u)^{1/2}, \\ &\text{(ii)} \quad Q(a_u) \sim u T(a_u)^{-1/2}. \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\text{(iii)} \quad T(a_{\alpha u}) \sim T(a_u), \\ &\text{(iv)} \quad Q(a_{\alpha u}) \sim Q(a_u), \\ &\text{(v)} \quad Q'(a_{\alpha u}) \sim Q'(a_u). \end{aligned} \tag{2.2}$$

(d) *If $\alpha > 1$ we have*

$$\left| \frac{a_{\alpha u}}{a_u} - 1 \right| \sim T(a_u)^{-1} \tag{2.3}$$

from which it follows in particular that $\forall \beta > 0$,

$$\frac{a_{\beta u}}{a_u} \to 1, \qquad u \to \infty. \tag{2.4}$$

(e) *For some* C_j, $j = 1, 2, 3$, *and* $s \geqslant r \geqslant C_3$

$$\left(\frac{s}{r}\right)^{C_1 T(r)} \leqslant \frac{Q(s)}{Q(r)} \leqslant \left(\frac{s}{r}\right)^{C_2 T(s)}. \tag{2.5}$$

(f) *There exists* $\varepsilon > 0$ *such that*

$$T(a_u) = O(u^{(2-\varepsilon)}). \tag{2.6}$$

Moreover, $\forall \delta > 0$

$$a_u = o(u^{\delta}), \qquad u \to \infty. \tag{2.7}$$

(g) *There exist* C_j, $j = 1, 2, 3$, *such that for* $v \geqslant u \geqslant C_3$

$$\left(\frac{a_v}{a_u}\right) \leqslant C_1 \left(\frac{v}{u}\right)^{C_2/T(a_u)}, \tag{2.8}$$

and

$$\left(\frac{a_v}{v}\right) \Big/ \left(\frac{a_u}{u}\right) \leqslant C_1 \left(\frac{v}{u}\right)^{(C_2/T(a_u))-1}. \tag{2.9}$$

In particular, given $\varepsilon > 0$, *we have for* $v \geqslant u \geqslant C_3$

$$\left(\frac{a_v}{a_u}\right) \leqslant C_1 \left(\frac{v}{u}\right)^{\varepsilon}, \tag{2.10}$$

$$\left(\frac{a_v}{v}\right) \Big/ \left(\frac{a_u}{u}\right) \leqslant C_1 \left(\frac{v}{u}\right)^{\varepsilon - 1}. \tag{2.11}$$

Proof. Firstly, (a)–(c) [(i)–(iii)], (2.3)–(2.6) are part of Lemmas 2.1 and 2.2 in [2]. The rest of (2.2) follows from (2.1). Relation (2.7) will follow using [(a)], as given $A > 0$

$$C(a_u)^A \leqslant Q(a_u) \sim uT(a_u)^{-1/2} \Rightarrow \frac{(a_u)^A}{u} \to 0, \qquad u \to \infty.$$

It remains to show (g). Now by (2.1) and then (2.5)

$$C_1 \frac{v}{u} \geqslant \frac{vT(a_v)^{-1/2}}{uT(a_u)^{-1/2}} \sim \frac{Q(a_v)}{Q(a_u)} \geqslant \left(\frac{a_v}{a_u}\right)^{C_2 T(a_u)},$$

which implies

$$\left(\frac{a_v}{a_u}\right) \leqslant C_3 \left(\frac{v}{u}\right)^{C_4/T(a_u)}.$$

So we have (2.8) and then (2.9)–(2.11) also follow. ▮

LEMMA 2.2. *Let $W \in \mathscr{E}_1$.*

(a) *Let $t > 0$ be small enough. Then there exists u such that*

$$t = \frac{a_u}{u}. \tag{2.12}$$

(b) *Let $\varepsilon > 0$. Then for u large enough*

$$\sigma\left(\frac{a_u}{u}\right) = a_{v(u)}, \tag{2.13}$$

where

$$u(1-\varepsilon) \leqslant v(u) \leqslant u.$$

(c) *Let $a > 1$. There exists $C_1, C_2 > 0$ such that for $s/a \leqslant t \leqslant s$ and $s \leqslant C_1$*

$$1 \leqslant \frac{\sigma(t)}{\sigma(s)} \leqslant 1 + \frac{C_2}{T(\sigma(t))}. \tag{2.14}$$

Further, for t small enough, we have for some $\varepsilon > 0$,

$$T(\sigma(t)) = O\left(\frac{\sigma(t)}{t}\right)^{2-\varepsilon}. \tag{2.15}$$

(d) *Recall the definition (1.9) and let $\beta \in (0, \infty)$. Then we have for some $C_1 > 0$ and $\forall x \in \mathbb{R}$*

$$\Phi_t^{\beta}(x) \geqslant C_1 T(\sigma(t))^{-\beta/2}. \tag{2.16}$$

Further if $m \leqslant n$ *and* $n, m \geqslant C_2$, *then*

$$\sup_{x \in \mathbb{R}} \frac{\Phi_{a_n/n}(x)}{\Phi_{a_m/m}(x)} \leqslant C_3 \sqrt{\log\left(2 + \frac{n}{m}\right)}, \tag{2.17}$$

for some $C_3 > 0$ *independent of* n, m, *and* x.

(e) *Given* $a > 1$, *there exists* $C_1 > 0$ *independent of* s, t, *and* x *such that for* $0 < s < C_1$ *and* $s/a \leqslant t \leqslant s$

$$\Phi_s(x) \sim \Phi_t(x), \qquad x \in \mathbb{R}. \tag{2.18}$$

(f) *Uniformly for* $n \geqslant 1$ *and* $x \in \mathbb{R}$,

$$\Phi_{a_n/n}(x) \sim \sqrt{\left|1 - \frac{|x|}{a_n}\right|} + T(a_n)^{-1/2}. \tag{2.19}$$

Further given $\beta > 0$, *we have for some* $C_1 > 0$ *and for all* $x \in \mathbb{R}$,

$$\Phi^{\beta}_{a_n/n}(x) \geqslant C_1 T(a_n)^{-\beta/2}. \tag{2.20}$$

Proof. Inequality (2.16) follows from the definition of Φ_t. Relations (2.12)–(2.14), and (2.17)–(2.19) are part of Lemmas 3.1 and 7.1 in [2]. Inequality (2.20) follows from (2.19). Finally to prove (2.15), we may by (2.12) put $t = a_u/u$ for some $u \geqslant u_o$. Then using Lemma 2.1(b), (2.13), and (2.6) gives for some $\varepsilon > 0$

$$T(\sigma(t)) \leqslant T(a_u) = O(u^{2-\varepsilon}) = O\left(\frac{\sigma(t)}{t}\right)^{2-\varepsilon}. \quad \blacksquare$$

We have an infinite-finite range inequality:

LEMMA 2.3. *Let* $W \in \mathscr{E}_1$, $0 < p \leqslant \infty$ *and* $s > 1$. *Then for some* C_1, C_2, $C_3 > 0$ *and* $\forall P \in \mathscr{P}_n$, $n \geqslant 1$,

$$\text{(a)} \quad \|PW\|_{L_p(\mathbb{R})} \leqslant C_1 \|PW\|_{L_p(-a_{sn}, a_{sn})}, \tag{2.21}$$

$$\text{(b)} \quad \|PW\|_{L_p(|x| \geqslant a_{sn})} \leqslant C_2 \exp[-C_3 n T(a_n)^{-1/2}] \, \|PW\|_{L_p(-a_{sn}, a_{sn})}. \tag{2.22}$$

Proof. This is Lemma 2.3 in [2]. $\blacksquare$

Note that (2.6) shows that for large n,

$$nT(a_n)^{-1/2} \geqslant n^{C_3}, \qquad \text{some} \quad C_3 > 0.$$

LEMMA 2.4. *Let $W \in \mathscr{E}_1$, $t \in (0, t_0)$ and $\beta > 0$. Put for $u \geqslant u_0$*

$$t = \frac{\beta a_u}{u}$$

and set

$$n := n(t) = \inf\left\{k: \frac{a_k}{k} \leqslant \frac{\beta a_u}{u}\right\}. \tag{2.23}$$

Then

$$\text{(a)} \quad \frac{a_n}{n} \leqslant \frac{\beta a_u}{u} < \frac{a_{n-1}}{n-1}, \tag{2.24}$$

$$\text{(b)} \quad \frac{a_n}{n} \leqslant \frac{\beta a_u}{u} < 2\frac{a_n}{n}, \tag{2.25}$$

$$\text{(c)} \quad u \sim n. \tag{2.26}$$

Proof. Inequality (2.24) follows from the definition of n. Inequality (2.25) follows from (2.24) as

$$a_{n-1} < a_n.$$

To show (2.26), we first show that $\exists \alpha > 0$ such that

$$u \leqslant \alpha n. \tag{2.27}$$

Suppose first that $u \geqslant n$. Using (2.24) and Lemma 2.1(g), there exists $C > 0$ such that

$$\frac{1}{\beta} \leqslant \frac{a_u}{u} \Big/ \frac{a_n}{n} \leqslant C\left(\frac{u}{n}\right)^{-1/2}$$

which implies (2.27). Suppose $u \leqslant n$. Then (2.27) follows with $\alpha = 1$. So it suffices to show that $\exists C_1 > 0$ such that

$$u \geqslant C_1 n.$$

If $n - 1 \geqslant u$ by (2.24) and Lemma 2.1(g), there exists $C_2 > 0$ such that

$$\beta \leqslant \frac{a_{n-1}}{n-1} \Big/ \frac{a_u}{u} \leqslant C_2 \left(\frac{n-1}{u}\right)^{-1/2}$$

which implies

$$u \geqslant C_3 n$$

for some $C_3 > 0$. Further, if $u \geqslant n-1$ we are done. ∎

We now present two lemmas on differences.

LEMMA 2.5. *Let* $W \in \mathscr{E}_1$.

(a) *Recall the difference operator* Δ_h^r *defined by* (1.8). *Then we have* $\forall x \in \mathbb{R}$, $\forall P \in \mathscr{P}_{r-1}$, $r \geqslant 1$, $\beta \in \mathbb{R}$, *and* $t > 0$

$$\begin{aligned} &\text{(i)} \quad \Delta^r_{h\Phi_t^\beta(x)} P(x) \equiv 0, \\ &\text{(ii)} \quad r!(h\Phi_t^\beta(x))^r = \Delta^r_{h\Phi_t^\beta(x)} x^r. \end{aligned} \tag{2.28}$$

(b) *Let* $L, s > 0$. *Then uniformly for* $u \geqslant 1$ *and* $|x|, |y| \leqslant a_{us}$ *such that*

$$|x-y| \leqslant L \frac{a_u}{u} \sqrt{1 - \left(\frac{|y|}{a_{us}}\right)} \qquad \textit{or} \quad |x-y| \leqslant L \frac{a_u}{u} T(a_u)^{-1/2},$$

we have

$$W(x) \sim W(y). \tag{2.29}$$

(c) *Let* $L, M > 0$. *For* $t \in (0, t_0)$, $|x|, |y| \leqslant \sigma(Mt)$ *such that*

$$|x-y| \leqslant Lt\Phi_t(x)$$

we have (2.29) *and*

$$\Phi_t(x) \sim \Phi_t(y). \tag{2.30}$$

Proof. This is Lemma 3.2 in [2].

LEMMA 2.6. *Let* $W \in \mathscr{E}_1$, $0 < \delta < 1$; $L, M > 0$ *and* $0 < p \leqslant \infty$.

(a) *Let* $s \in (0, 1)$ *and* $[a, b]$ *be contained in one of the ranges*

$$|x| \leqslant \sigma(t) \left[1 - \left(\frac{s}{2\delta\sigma(t)} \right)^2 \right] \tag{2.31}$$

or

$$|x| \geqslant \sigma(t) \left[1 + \left(\frac{s}{2\delta\sigma(t)} \right)^2 \right]. \tag{2.32}$$

Then

$$\int_a^b |f(x \pm s\Phi_t(x))|\,dx \leqslant \frac{2}{1-\delta} \int_{\bar{a}}^{\bar{b}} |f(x)|\,dx, \tag{2.33}$$

where

$$\begin{Bmatrix} \bar{a} \\ \bar{b} \end{Bmatrix} := \begin{Bmatrix} \inf \\ \sup \end{Bmatrix} \{x \pm s\Phi_t(x) : x \in [a, b]\}. \tag{2.34}$$

(b) *Let* $r \geqslant 1$, $t \in (0, 1/M)$, $h \in (0, Mt)$, *and* $[a, b]$ *be as above with* $s = Mrt$. *Define* $\bar{a}$ *and* $\bar{b}$ *by* (2.34) *with* $s = Mrt$. *Assume moreover that*

$$[a, b] \subseteq [-\sigma(Lt), \sigma(Lt)]. \tag{2.35}$$

Then for some $C \neq C(a, b, t, g)$

$$\begin{aligned} \|\Delta^r_{h\Phi_t(x)}(g, x, \mathbb{R})\, W(x)\|_{L_p[a,b]} &\leqslant C \inf_{P \in \mathscr{P}_{r-1}} \|W(g-P)\|_{L_p[\bar{a}, \bar{b}]} \\ &\leqslant C\, \|Wg\|_{L_p[\bar{a}, \bar{b}]}. \end{aligned} \tag{2.36}$$

Proof. (a) Define $\kappa = \pm 1$ and $u(x) := x + \kappa s\Phi_t(x)$.

We shall assume that $[a, b]$ is contained in the range (2.31) and also $a \geqslant 0$. The case where $a < 0$ is similar, as is the case when $[a, b]$ is contained in the range (2.32). Then for $x \in [a, b]$,

$$u'(x) = 1 + \frac{\kappa s}{2\sqrt{1-(x/\sigma(t))}} \left(-\frac{1}{\sigma(t)}\right) \geqslant 1 - \delta,$$

by (2.31). Hence u is increasing in $[a, b]$ and writing $v := u(x)$ gives

$$\begin{aligned} \int_a^b |f(x \pm s\Phi_t(x))|\,dx &= \int_a^b |f(u(x))|\,dx \\ &= \int_{u(a)}^{u(b)} |f(v)| \frac{dx}{du}\,dv, \qquad v = u(x) \\ &\leqslant \frac{1}{1-\delta} \int_{u(a)}^{u(b)} |f(v)|\,dv \\ &= \frac{1}{1-\delta} \int_{\bar{a}}^{\bar{b}} |f(x)|\,dx \end{aligned}$$

in this case. The extra 2 in (2.33) takes care of having to split $[a, b]$ into two intervals if $a < 0 < b$.

(b) Now recall that we have

$$W(x)\,\Delta^r_{h\Phi_t(x)}(g(x)) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i\, W(x)\, g\left(x + \left(\frac{r}{2} - i\right) h\Phi_t(x)\right).$$

Also (2.29) gives

$$W(x) \sim W\left(x + \left(\frac{r}{2} - i\right) h\Phi_t(x)\right)$$

uniformly in i and for $|x| \leqslant \sigma(Lt)$ and $h \leqslant Mt$. Thus we obtain from part [(a)]

$$\begin{aligned} \| W(x)\Delta^r_{h\Phi_t(x)}(g(x))\|_{L_p[a,\,b]} &\leqslant C \sup_{0\leqslant i\leqslant r} \int_a^b |gW|^p \left(x + \left(\frac{r}{2} - i\right) h\Phi_t(x)\right) dx \\ &\leqslant \frac{2C}{1-\delta} \int_{\bar{a}}^{\bar{b}} |gW|^p\,(x)\, dx. \end{aligned}$$

Note that for $0 \leqslant i \leqslant r$, (2.31) with $s = Mrt$ gives

$$|x| \leqslant \sigma(t) \left(1 - \left[\frac{Mrt}{2\delta\sigma(t)}\right]^2\right) \leqslant \sigma(t) \left(1 - \left[\frac{ih}{4\delta\sigma(t)}\right]^2\right)$$

so the range restrictions of (a) are satisfied.

Finally note that by (2.28) for $P \in \mathscr{P}_{r-1}$,

$$\Delta^r_{h\Phi_t(x)}(P, x, \mathbb{R}) \equiv 0.$$

Hence

$$\begin{aligned} \|\Delta^r_{h\Phi_t(x)}(g, x, \mathbb{R})\, W(x)\|_{L_p[a,\,b]} &= \|\Delta^r_{h\Phi_t(x)}(g - P, x, \mathbb{R})\, W(x)\|_{L_p[a,\,b]} \\ &\leqslant C\, \|(g-P)\, W\|_{L_p[\bar{a},\,\bar{b}]}. \end{aligned}$$

It remains to take the infimum over P. ∎

3. A MARKOV–BERNSTEIN INEQUALITY

In this section, we prove an extension of the Markov–Bernstein inequality (1.20).

THEOREM 3.1. *Let $W \in \mathscr{E}_1$ and assume (1.20). Let $0 < p \leqslant \infty$ and define for $n \geqslant 1$,*

$$\Psi_n(x) := \left(1 - \left(\frac{x}{a_n}\right)^2\right)^2 + T(a_n)^{-2}, \qquad x \in \mathbb{R}. \tag{3.1}$$

Then for $n \geqslant C_1$, $0 \leqslant l \leqslant n$, and $\forall P \in \mathscr{P}_n$ we have

$$\|P^{(l+1)} \Psi_n^{(l+1)/4} W\|_{L_p(\mathbb{R})} \leqslant C_2 \left\{\frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{1/2}\right\} \|P^{(l)} \Psi_n^{l/4} W\|_{L_p(\mathbb{R})} \tag{3.2}$$

$$\leqslant C_3 \frac{n}{a_n} [l+1] \, \|P^{(l)} \Psi_n^{l/4} W\|_{L_p(\mathbb{R})}. \tag{3.3}$$

Here $C_j \neq C_j(n, l, P)$, $j = 2, 3$.

We remark that (3.2) and (3.3) will hold with constants depending on l if we replace $\Psi_n^{1/4}$ by $\Phi_{a_n/n}$.

More precisely,

$$\|P^{(l+1)} \Phi_{a_n/n}^{l+1} W\|_{L_p(\mathbb{R})} \leqslant C_4^l \left\{\frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{1/2}\right\} \|P^{(l)} \Phi_{a_n/n}^{l} W\|_{L_p(\mathbb{R})} \tag{3.4}$$

$$\leqslant C_5^l \frac{n}{a_n} [l+1] \, \|P^{(l)} \Phi_{a_n/n}^{l} W\|_{L_p(\mathbb{R})}, \tag{3.5}$$

where $C_j \neq C_j(n, P)$, $j = 4, 5$.

We need several lemmas.

LEMMA 3.2. *Let $s > 1$ and $n \geqslant C_1$. Then there exist polynomials R of degree $o(n)$ such that uniformly for $|x| \leqslant a_{sn}$*

$$R(x) \sim \Phi_{a_n/n}(x) \sim \Psi_n^{1/4}(x) \tag{3.6}$$

and

$$|R'(x)/R(x)| \leqslant \frac{C_1}{a_n} \Psi_n^{-1/2}(x). \tag{3.7}$$

Proof. Let

$$u(x) := (1 - x^2)^{-3/4}, \qquad x \in [-1, 1]$$

be the ultraspherical weight on $(-1, 1)$ and let $\lambda_n(u, x)$ be the Christoffel function corresponding to u satisfying

$$\lambda_n^{-1}(u, x) \in \mathscr{P}_{2n-2}.$$

Then it is known [25, p. 36], that given $A>0$ we have uniformly in n and $|x|\leqslant 1-(A/n^2)$

$$\lambda_n(u,x)\sim\frac{1}{n}(1-x^2)^{-1/4} \tag{3.8}$$

and

$$|\lambda_n'(u,x)|\leqslant\frac{C_1}{n}(1-x^2)^{-5/4}. \tag{3.9}$$

Now choose $m:=m(n)$ = the largest integer $\leqslant T(a_n)^{-1/2}$ and put

$$R(x):=\frac{1}{m^2}\lambda_m^{-2}\left(u,\frac{x}{a_{2sn}}\right),\qquad x\in[-a_{sn},a_{sn}].$$

Then by (2.6), R has degree $o(n)$ and by (2.3), (2.6), (2.19), (3.1), and (3.8) we have uniformly for $|x|\leqslant a_{sn}$,

$$R(x)\sim\Phi_{a_n/n}(x)\sim\Psi_n^{1/4}(x).$$

To prove (3.7), we observe much as in [22, p. 228] that

$$\left|\lambda_n^{-1}\left(u,\frac{x}{a_{2sn}}\right)'\right|=\frac{|\lambda_n'(u,x/a_{2sn})|}{a_{2sn}\lambda_n^2(u,x/a_{2sn})}, \tag{3.10}$$

so that by (3.8), (3.9), and the definition of R we have uniformly for $|x|\leqslant a_{sn}$,

$$\begin{aligned}|R'(x)/R(x)|&\leqslant\frac{C_2}{a_n}\left(1-\left(\frac{x}{a_{2sn}}\right)^2\right)^{-1}\\&\leqslant\frac{C_3}{a_n}\Psi_n(x)^{-1/2}.\quad\blacksquare\end{aligned}$$

Our next lemma is an infinite-finite range inequality:

LEMMA 3.3. *Let $W\in\mathscr{E}_1$. Let $0<p\leqslant\infty$, $s>1$, and Ψ_n be as in (3.1). Then for $n\geqslant C_1$, $\forall P\in\mathscr{P}_n$, and $0\leqslant l\leqslant n$ we have*

$$\|PW\Psi_n^{l/4}\|_{L_p(\mathbb{R})}\leqslant C_1\,\|PW\Psi_n^{l/4}\|_{L_p(|x|\leqslant a_{3sn})}. \tag{3.11}$$

Moreover,

$$\|PW\Psi_n^{l/4}\|_{L_p(|x|\geqslant a_{3sn})}\leqslant C_2\exp[-C_3n^{C_4}]\,\|PW\Psi_n^{l/4}\|_{L_p(|x|\leqslant a_{3sn})}. \tag{3.12}$$

Here, $C_j\neq C_j(n,P,l)$, $j=1,2$.

We recall that (2.6) shows that for large n,

$$nT(a_n)^{-1/2} \geqslant n^{C_3}. \tag{3.13}$$

Proof. First note that by (2.20) and the definition of Ψ_n, given $\beta > 0$ we have

$$\Psi_n^{\beta/4}(x) \geqslant T(a_n)^{-\beta/2}, \qquad x \in \mathbb{R}. \tag{3.14}$$

Now write $l = 4j + k$, $0 \leqslant k < 3$. Then for some $0 < \alpha \leqslant 3$ and C_1 depending on k we have

$$\begin{aligned} \|PW\Psi_n^{l/4}\|_{L_p(|x| \geqslant a_{3sn})} &= \|PW\Psi_n^j \Psi_n^{k/4}\|_{L_p(|x| \geqslant a_{3sn})} \\ &\leqslant C_1 \|PW\Psi_n^j x^\alpha\|_{L_p(|x| \geqslant a_{3sn})}. \end{aligned} \tag{3.15}$$

Now $Px^\alpha \Psi_n^j$ is a polynomial of degree $\leqslant n + l + 3 \leqslant 3n$ so by (2.22), we may continue (3.15) as

$$\begin{aligned} &\leqslant C_2 \exp[-C_3 nT(a_n)^{-1/2}] \|PWx^\alpha \Psi_n^j\|_{L_p(|x| \leqslant a_{3sn})} \\ &\leqslant C_4 \exp[-C_3 nT(a_n)^{-1/2}]\, a_n^\alpha T(a_n)^{k/2} \|PW\Psi_n^{j+k/4}\|_{L_p(|x| \leqslant a_{3sn})} \quad \text{(by (3.14))} \\ &\leqslant C_5 \exp[-C_6 nT(a_n)^{-1/2}] \|PW\Psi_n^{l/4}\|_{L_p(|x| \leqslant a_{3sn})} \end{aligned}$$

by (2.7) and (3.13). ∎

We can now give:

Proof of Theorem 3.1. We prove (3.2). Then (3.3) will follow by (2.6). Inequalities (3.4) and (3.5) will follow as

$$\Psi_n^{1/4}(x) \sim \Phi_{a_n/n}(x), \qquad x \in \mathbb{R}.$$

Put $s > 1$ and write $l = 4j + k$, $0 \leqslant k \leqslant 3$. Put $Q := P^{(l)}$. Then

$$\begin{aligned} J := \|P^{(l+1)} W\Psi_n^{(l+1)/4}\|_{L_p(|x| \leqslant a_{3sn})} &= \|Q' W\Psi_n^{(l+1)/4}\|_{L_p(|x| \leqslant a_{3sn})} \\ &= \|Q' W\Psi_n^{j+(k+1)/4}\|_{L_p(|x| \leqslant a_{3sn})}. \end{aligned}$$

Choose by Lemma 3.2, R of degree $o(n)$ such that

$$R(x) \sim \Psi_n^{1/4}(x)$$

and

$$|R'(x)/R(x)| \leqslant \frac{C_1}{a_n} \Psi_n^{-1/2}(x)$$

uniformly for $|x| \leqslant a_{3sn}$.

Then continue this estimate as

$$J \leqslant C_2 \| Q' W \Psi_n^j R^k \Psi_n^{1/4} \|_{L_p(|x| \leqslant a_{3sn})},$$

where C_2 depends only on k. This in turn can be continued as

$$\begin{aligned}
&\leqslant C_2 \|(Q\Psi_n^j R^k)' \Psi_n^{1/4} W\|_{L_p(|x| \leqslant a_{3sn})} + C_2 \|(\Psi_n^j)' R^k Q \Psi_n^{1/4} W\|_{L_p(|x| \leqslant a_{3sn})} \\
&\quad + C_2 \|\Psi_n^j (R^k)' Q \Psi_n^{1/4} W\|_{L_p(|x| \leqslant a_{3sn})} \\
&= T_1 + T_2 + T_3.
\end{aligned}$$

We begin with the estimation of T_1.

Note that $Q\Psi_n^j R^k$ is a polynomial of degree $\leqslant n + l + o(n) \leqslant 3n$. Thus, we can write

$$\begin{aligned}
T_1 &\leqslant C_3 \frac{n}{a_n} \|Q\Psi_n^j R^k W\|_{L_p(\mathbb{R})} && \text{(by (1.20))} \\
&\leqslant C_4 \frac{n}{a_n} \|Q\Psi_n^j R^k W\|_{L_p(|x| \leqslant a_{3sn})} && \text{(by (2.21))} \\
&\leqslant C_5 \frac{n}{a_n} \|Q\Psi_n^{j+k/4} W\|_{L_p(|x| \leqslant a_{3sn})} \\
&\leqslant C_5 \frac{n}{a_n} \|P^{(l)} \Psi_n^{l/4} W\|_{L_p(\mathbb{R})}. && (3.16)
\end{aligned}$$

Next we estimate T_2.

Note that for $|x| \leqslant a_{sn}$ and by straightforward differentiation, (2.3) gives

$$|(\Psi_n^j)'|(x) \leqslant C_6 \Psi_n(x)^{j-(1/2)} \frac{j}{a_n}.$$

Thus

$$\begin{aligned}
T_2 &\leqslant C_7 \frac{j}{a_n} \|P_n^{(l)} \Psi_n^{j-(1/2)} \Psi_n^{k/4 + 1/4} W\|_{L_p(|x| \leqslant a_{3sn})} \\
&\leqslant C_7 \frac{j}{a_n} \|P_n^{(l)} \Psi_n^{l/4 - 1/4} W\|_{L_p(|x| \leqslant a_{3sn})} \\
&\leqslant C_8 \frac{lT(a_n)^{1/2}}{a_n} \|P_n^{(l)} \Psi_n^{l/4} W\|_{L_p(\mathbb{R})} && (3.17)
\end{aligned}$$

by (3.14).

It remains to estimate T_3.

Write

$$T_3 \leqslant C_9 k \,\|\Psi_n^j R^{k-1} R' Q \Psi_n^{1/4} W\|_{L_p(|x| \leqslant a_{3sn})}$$
$$\leqslant \frac{C_{10} k}{a_n} \|\Psi_n^j \Psi_n^{k-1/4} Q W\|_{L_p(|x| \leqslant a_{3sn})} \qquad \text{by (3.7)}$$
$$\leqslant C_{10} \frac{l T(a_n)^{1/2}}{a_n} \|P_n^{(l)} \Psi_n^{l/4} W\|_{L_p(\mathbb{R})} \tag{3.18}$$

as in the estimation of T_2.

Combining (3.16), (3.17), and (3.18) gives

$$J \leqslant C_{11} \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{1/2} \right\} \|P^{(l)} W \Psi_n^{l/4}\|_{L_p(\mathbb{R})}, \tag{3.19}$$

where $C_{11} \neq C_{11}(n, P, l)$.

Finally by (3.11), (3.19) becomes

$$\|P^{(l+1)} W \Psi_n^{l+1/4}\|_{L_p(\mathbb{R})} \leqslant C_{12} \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{1/2} \right\} \|P^{(l)} W \Psi_n^{l/4}\|_{L_p(\mathbb{R})}$$

as required where $C_{12} \neq C_{12}(n, P, l)$. ∎

4. APPROXIMATION OF POLYNOMIALS OF DEGREE $\leqslant n$ BY THOSE OF DEGREE $\leqslant r-1$

In this section, we obtain a crucial inequality introduced in a related context in [8], in order to obtain an upper bound for our modulus in terms of our realization-functional. The main idea is to approximate polynomials of degree $\leqslant n$ by polynomials of degree $\leqslant r-1$. Here $n \geqslant n_0$ and $r \geqslant 1$.

We prove:

THEOREM 4.1. *Let $W \in \mathscr{E}_1$ and assume* (1.20). *Let $r \geqslant 1$, $L > 0$, $0 < p \leqslant \infty$, $P_n \in \mathscr{P}_n$, and $n \geqslant C$. Set*

$$P(x) := P_n(x) - \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_{r-1}} \cdots \int_{a_{Ln}}^{u_1} P_n^{(r)}(u_o)\, du_o \cdots du_{r-1} \in \mathscr{P}_{r-1}. \tag{4.1}$$

Then, $\exists C_1 > 0$, $C_1 \neq C_1(n, P_n, P)$ such that

$$\|W(P_n - P)\|_{L_p[a_{Ln}, \infty)} \leqslant C_1 \left(\frac{a_n}{n}\right)^r \|W P_n^{(r)} \Phi_{a_n/n}^r\|_{L_p(\mathbb{R})}. \tag{4.2}$$

We break the proof down into several steps. We begin with:

LEMMA 4.2. *Let $W \in \mathscr{E}_1$, $1 \leqslant p \leqslant \infty$. Then for $n \geqslant C$ and $\forall g \in L_p[a_{Ln}, \infty)$, $\exists C_1 > 0$, $C_1 \neq C_1(g, n)$ such that*

$$\left\| W(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln}, \infty)} \leqslant \frac{a_n}{nT(a_n)^{1/2}} \|gW\|_{L_p[a_{Ln}, \infty)}. \tag{4.3}$$

Proof. We notice that

$$W(x)^{1/2} \int_{t}^{x} W(u)^{-1/2} Q'(u)\, du = 2\left[1 - \left[\frac{W(x)}{W(t)}\right]^{1/2}\right] \leqslant 2 \tag{4.4}$$

as $t \leqslant x$.

Next, notice that for $u \geqslant a_{Ln}$, and n large enough, we have by Lemma 2.1

$$Q'(u) \geqslant CQ'(a_{Ln}) \sim \frac{nT(a_n)^{1/2}}{a_n}, \tag{4.5}$$

so that for $x \geqslant a_{Ln}$

$$\begin{aligned} &\frac{a_n}{nT(a_n)^{1/2}} W(x)^{1/2} \int_{a_{Ln}}^{x} |gW(u)|\, Q'(u) W^{-1/2}(u)\, du \\ &\quad \geqslant C_1 W(x)^{1/2} \int_{a_{Ln}}^{x} |gW(u)^{1/2}|\, du \geqslant W(x) \left| \int_{a_{Ln}}^{x} g(u)\, du \right|. \end{aligned} \tag{4.6}$$

Now recalling Jensen's Inequality for integrals

$$\left| \int f\, d\mu \right|^p \leqslant \left(\int |f|^p\, d\mu \right) \left(\int d\mu \right)^{p-1}$$

valid for μ measurable functions f and non-negative measures μ, gives:

Case 1. $p = \infty$. Here (4.6) gives for $x \geqslant a_{Ln}$

$$\begin{aligned} W(x) \left| \int_{a_{Ln}}^{x} g(u)\, du \right| &\leqslant \frac{a_n}{nT(a_n)^{1/2}} W(x)^{1/2} \|gW\|_{L_\infty[a_{Ln}, \infty)} \int_{a_{Ln}}^{x} Q'(u) W^{-1/2}(u)\, du \\ &\leqslant C_2 \frac{a_n}{nT(a_n)^{1/2}} \|gW\|_{L_\infty[a_{Ln}, \infty)} \qquad \text{(by (4.4))}. \end{aligned}$$

Case 2. $1 \leqslant p < \infty$. Here

$$\left\| W(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln},\, \infty)}$$

$$\leqslant \frac{a_n}{nT(a_n)^{1/2}} \left[\int_{a_{Ln}}^{\infty} \left[W(x)^{1/2} \int_{a_{Ln}}^{x} |gW(u)|\, Q'(u)\, W^{-1/2}(u)\, du \right]^p dx \right]^{1/p}$$

$$\leqslant C_3 \frac{a_n}{nT(a_n)^{1/2}} \left[\int_{a_{Ln}}^{\infty} 2^{p-1} W(x)^{1/2} \int_{a_{Ln}}^{x} |gW(u)|^p\, Q'(u)\, W^{-1/2}(u)\, du\, dx \right]^{1/p}$$

by Jensen's Inequality, with $d\mu = W(x)^{1/2}\, Q'(u)\, W(u)^{-1/2}$ on $[a_{Ln}, x]$ and $\int d\mu \leqslant 2$ (see (4.4)).

Then

$$\int_{a_{Ln}}^{\infty} W(x)^{1/2} \int_{a_{Ln}}^{x} |gW(u)|^p\, Q'(u)\, W^{-1/2}(u)\, du\, dx$$

$$= \int_{a_{Ln}}^{\infty} |gWu|^p \left[\int_{u}^{\infty} W(x)^{1/2}\, Q'(u)\, dx \right] W^{-1/2}(u)\, du$$

$$\leqslant C_4 \int_{a_{Ln}}^{\infty} |gW(u)|^p \left[\int_{u}^{\infty} W(x)^{1/2}\, Q'(x)\, dx \right] W^{-1/2}(u)\, du \qquad (\text{as } x > u)$$

$$\leqslant C_5 \, \|gW\|^p_{L_p[a_{Ln},\, \infty)}. \quad \blacksquare$$

We are now in the position to give:

Proof of Theorem 4.1 *for* $1 \leqslant p \leqslant \infty$. We will repeatedly make use of (2.20):

$$\Phi_{a_n/n}(x) \geqslant CT(a_n)^{-1/2}, \qquad \forall x \in \mathbb{R}. \tag{4.7}$$

Firstly, if $r = 1$, Lemma 4.2 with $g = P'_n$ gives

$$\left\| W(x) \int_{a_{Ln}}^{x} P'_n(u_o)\, du_o \right\|_{L_p[a_{Ln},\, \infty)} \leqslant C_1 \frac{a_n}{nT(a_n)^{1/2}} \|P'_n W\|_{L_p(\mathbb{R})}$$

$$\leqslant C_2 \frac{a_n}{n} \|P'_n \Phi_{a_n/n}(x)\, W\|_{L_p(\mathbb{R})} \qquad (\text{by } (4.7)).$$

Now apply (4.1). If $r = 2$, we apply Lemma 4.2 with

$$g(u_1) = \int_{a_{nL}}^{u_1} P^{(2)}(u_o)\, du_o$$

to give

$$
\begin{aligned}
\left\| W(x) \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_1} P_n^{(2)}(u_o)\, du_o\, du_1 \right\|_{L_p[a_{Ln},\, \infty)}
&= \left\| W(x) \int_{a_{Ln}}^{x} g(u_1)\, du_1 \right\|_{L_p[a_{Ln},\, \infty)} \\
&\leqslant C_3 \frac{a_n}{nT(a_n)^{1/2}} \| gW \|_{L_p[a_{Ln},\, \infty)} \\
&= C_3 \frac{a_n}{nT(a_n)^{1/2}} \left\| W \int_{a_{nL}}^{u_1} P_n^{(2)}(u_o)\, du_o \right\|_{L_p[a_{Ln},\, \infty)} \\
&\leqslant C_4 \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^2 \| P_n^{(2)} W \|_{L_p(\mathbb{R})} \\
&\leqslant C_5 \left(\frac{a_n}{n} \right)^2 \| P_n^{(2)} \Phi_{a_n/n}^2(x)\, W \|_{L_p(\mathbb{R})}.
\end{aligned}
$$

Applying now (4.1) and an induction argument on r gives the result. ▮

We now treat the more complicated case, $0 < p < 1$. For this case we need two lemmas.

LEMMA 4.3. *Let* $W \in \mathscr{E}_1$ *and assume* (1.20). *Let* $0 < p < 1$, $r \geqslant 1$, $R_n \in \mathscr{P}_n$, $R \in \mathscr{P}_{r-1}$, *and* $n \geqslant C$. *Set for* $x \in \mathbb{R}$, *and* $L > 0$

$$g_n(x) := (R_n - R)(x)$$

and

$$J_n(x) := \left\| |g_n' W(u)|^{1-p} \left(\frac{W(x)}{W(u)} \right)^{1/2} \right\|_{L_\infty[a_{Ln},\, x]}^{p/(1-p)}. \tag{4.8}$$

Then

$$
\begin{aligned}
\int_{a_{Ln}}^{\infty} J_n(x)\, dx \leqslant C_1 \Bigg[& \sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-1)p} \| W(R_n^{(j)} - R^{(j)}) \|_{L_\infty[a_{Ln},\, \infty)}^p \\
& + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-1)p} \| W R_n^{(r)} \|_{L_\infty(\mathbb{R})}^p \Bigg].
\end{aligned} \tag{4.9}
$$

Here $C_1 \neq C_1(n, R_n, R)$.

Proof. Write

$$J_n(x) = \left\| |g_n' W(u)|^p \left(\frac{W(x)}{W(u)} \right)^{p/2(1-p)} \right\|_{L_\infty[a_{Ln},\, x]}$$

and set

$$\tau := \frac{\delta a_n}{nT(a_n)^{1/2}},$$

where $\delta > 0$ is chosen small enough so that for $n \geqslant 1$ and $\forall S \in \mathscr{P}_n$,

$$\|S' W\|_{L_p(\mathbb{R})} \leqslant (2\delta^{-1}) \frac{nT(a_n)^{1/2}}{a_n} \|SW\|_{L_p(\mathbb{R})}. \tag{4.10}$$

(See (1.20) and (2.20).)

Now given $x \geqslant a_{Ln}$, we set

$$k_o := k_o(x) = \max\{k_o \colon x - (k+1)\,\tau \geqslant a_{Ln}\}$$

and write

$$J_n(x) \leqslant I_1 + I_2,$$

where

$$I_1 := \max_{0 \leqslant k \leqslant k_o} \left\| |g_n' W|^p\,(u) \left(\frac{W(x)}{W(u)}\right)^{p/2(1-p)} \right\|_{L_\infty[x-(k+1)\,\tau,\, x-k\tau]} \tag{4.11}$$

and

$$I_2 := \left\| |g_n' W(u)|^p \left(\frac{W(x)}{W(u)}\right)^{p/2(1-p)} \right\|_{L_\infty[a_{Ln},\, x-(k_o+1)\,\tau]}. \tag{4.12}$$

First we observe that for $u \in [x-(k+1)\,\tau,\ x-k\tau]$

$$\frac{W(x)}{W(u)} \leqslant \exp(Q(x-k\tau) - Q(x)).$$

Further, as $x - k\tau \geqslant a_{Ln} > 0$

$$\begin{aligned} Q(x) - Q(x-k\tau) &\geqslant C_1 k\tau Q'(x-k\tau) \geqslant C_2 k\tau Q'(a_{Ln}) \geqslant C_3 \frac{nT(a_n)^{1/2}\,\delta a_n k}{a_n nT(a_n)^{1/2}} \\ &= C_3 k\delta \end{aligned}$$

by (2.1) of Lemma 2.1. So

$$\left(\frac{W(x)}{W(u)}\right)^{p/2(1-p)} \leqslant \alpha^k, \qquad u \in [x-(k+1)\,\tau,\, x-k\tau],$$

where $\alpha \in (0, 1)$ is independent of x, u, k. Thus we may write

$$
\begin{aligned}
I_1 + I_2 &\leqslant \max_{0 \leqslant k \leqslant k_o} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \\
&\quad + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]} \\
&\leqslant \sum_{k=0}^{k_o(x)} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{a_{Ln}}^{\infty} J_n(x)\, dx &= \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} J_n(x)\, dx \\
&\leqslant \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \Bigg[\sum_{k=0}^{k_o(x)} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \\
&\quad + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]}\, dx \Bigg].
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]}\, dx \\
&\qquad = \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} \|g_n' W\|^p_{L_\infty[x,\, x+\tau]}\, dx
\end{aligned}
$$

and since

$$
x \in [a_{Ln} + (m-k-1)\tau,\, a_{Ln} + (m-k)\tau] \Rightarrow m \geqslant k_o \geqslant m-1,
$$

we have

$$
\begin{aligned}
\int_{a_{Ln}}^{\infty} J_n(x)\, dx &\leqslant \sum_{m=0}^{\infty} \Bigg[\sum_{k=0}^{m-1} \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} \alpha^k \|g_n' W\|^p_{L_\infty[x,\, x+\tau]}\, dx \\
&\quad + 2\alpha^{m-1} \int_{a_{Ln}}^{a_{Ln}+\tau} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x]}\, dx \Bigg] \\
&\leqslant \sum_{s=0}^{\infty} \Bigg[\int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau} \|g_n' W\|^p_{L_\infty[x,\, x+\tau]}\, dx \Bigg(\sum_{\substack{(m,k) \\ s=m-k-1}} \alpha^k \Bigg) \Bigg] \\
&\quad + 2 \int_{a_{Ln}}^{a_{Ln}+\tau} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x]} \frac{1}{\alpha(1-\alpha)}\, dx \\
&\leqslant C_4 [I_3 + I_4].
\end{aligned}
$$

Here

$$I_3 := \sum_{s=0}^{\infty} \int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau} \|g_n' W\|_{L_\infty[x, x+\tau]}^p \, dx \tag{4.13}$$

and

$$I_4 := \int_{a_{Ln}}^{a_{Ln}+\tau} \|g_n' W\|_{L_\infty[a_{Ln}, x]}^p \, dx \tag{4.14}$$

We begin by estimating I_3. Observe that g_n' is a polynomial of degree $\leqslant n-1$ for $u \in [x, x+\tau]$, so expanding it in a Taylor series about x gives

$$\begin{aligned}
|g_n'(u)|^p &= \left| \sum_{j=1}^{n} \frac{g_n^{(j)}(x)(u-x)^{j-1}}{(j-1)!} \right|^p \\
&\leqslant \sum_{j=1}^{n} |g_n^{(j)}(x)|^p \, \tau^{(j-1)p} \\
&\qquad \text{(by the inequality, } (a+b)^\alpha \leqslant a^\alpha + b^\alpha,\ 0<\alpha<1,\ a, b \in \mathbb{R}) \\
&\leqslant \sum_{j=1}^{r-1} |R_n^{(j)}(x) - R^{(j)}(x)|^p \, \tau^{(j-1)p} + \sum_{j=r}^{n} |R_n^{(j)}(x)|^p \, \tau^{(j-1)p}.
\end{aligned}$$

Thus using

$$W(u) \leqslant W(x), \qquad u \in [x, x+\tau], \tag{4.15}$$

the definition of τ, and (4.10) gives

$$\begin{aligned}
I_3 &\leqslant C_5 \left[\sum_{j=1}^{r-1} \|(R_n^{(j)} - R^{(j)}) W\|_{L_p[a_{Ln}, \infty)}^p \, \tau^{(j-1)p} \right. \\
&\qquad \left. + \tau^{(r-1)p} \sum_{j=r}^{n} \|R_n^{(j)} W\|_{L_p[a_{Ln}, \infty)}^p \, \tau^{(j-r)p} \right] \\
&\leqslant C_6 \left[\sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-1)p} \|(R_n^{(j)} - R^{(j)}) W\|_{L_p[a_{Ln}, \infty)}^p \right. \\
&\qquad \left. + \tau^{(r-1)p} \sum_{j=r}^{n} \left(\frac{\tau n T(a_n)^{1/2}}{2\delta a_n} \right)^{(j-r)p} \|R_n^{(r)} W\|_{L_p(\mathbb{R})}^p \right] \\
&\leqslant C_7 \left[\sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-1)p} \|(R_n^{(j)} - R^{(j)}) W\|_{L_p[a_{Ln}, \infty)}^p \right. \\
&\qquad \left. + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-1)p} \|R_n^{(r)} W\|_{L_p(\mathbb{R})}^p \right].
\end{aligned} \tag{4.16}$$

To estimate I_4 we proceed in a similar way to that of I_3, except that we use (2.29) instead of (4.15), which we may do in view of the definition of τ, (2.3), and (2.6). Combining our estimates for I_3 and I_4 gives the lemma. ▮

LEMMA 4.4. *Let $W \in \mathscr{E}_1$ and assume* (1.20). *Let $0 < p < 1$, $r \geqslant 1$, $L > 0$, $R_n \in \mathscr{P}_n$, $R \in \mathscr{P}_{r-1}$ satisfying,*

$$(R_n - R)(a_{Ln}) = 0.$$

Then for $n \geqslant C$ there exists $C_1 \neq C_1(n, R_n, R)$ such that

$$\begin{aligned}
&\| W(R_n - R) \|_{L_p[a_{Ln}, \infty)} \\
&\quad \leqslant C_1 \left[\left[\left(\frac{a_n}{nT(a_n)^{1/2}} \right) \| W(R_n' - R') \|^p_{L_p[a_{Ln}, \infty)} \right] \right. \\
&\qquad \times \left[\sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-1)(1-p)} \| (R_n^{(j)} - R^{(j)}) W \|^{1-p}_{L_p[a_{Ln}, \infty)} \right] \\
&\qquad \left. + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-1)(1-p)} \| R_n^{(r)} W \|^{1-p}_{L_p(\mathbb{R})} \right].
\end{aligned} \tag{4.17}$$

Proof. Set

$$g_n(x) := (R_n - R)(x)$$

satisfying $g_n(a_{Ln}) = 0$ and write

$$g_n(x) = \int_{a_{Ln}}^{x} g_n'(u)\, du.$$

Then

$$\begin{aligned}
\Delta &= \| W(R_n - R) \|_{L_p[a_{Ln}, \infty)} = \| W g_n \|_{L_p[a_{Ln}, \infty)} \\
&= \left[\int_{a_{Ln}}^{\infty} \left| \int_{a_{Ln}}^{x} g_n' W(u) \frac{W(x)}{W(u)}\, du \right|^p dx \right]^{1/p} \\
&\leqslant \left[\int_{a_{Ln}}^{\infty} \left\| |g_n' W(u)|^{1-p} \left(\frac{W(x)}{W(u)} \right)^{1/2} \right\|^p_{L_\infty[a_{Ln}, \infty)} \right. \\
&\qquad \left. \times \left(\int_{a_{Ln}}^{x} |g_n' W(u)|^p \left(\frac{W(x)}{W(u)} \right)^{1/2} du \right)^p dx \right]^{1/p}.
\end{aligned} \tag{4.18}$$

Now apply Hölder's Inequality with $r = 1/1 - p$, $\sigma = 1/p$ satisfying $r^{-1} + \sigma^{-1} = 1$ to give

$$\Delta \leqslant I_1 I_2,$$

where

$$I_1 := \left(\int_{a_{Ln}}^{\infty} \left\| |g_n' W(u)|^{1-p} \left(\frac{W(x)}{W(u)} \right)^{1/2} dx \right\|_{L_\infty[a_{Ln}, \infty)}^{p/1-p} \right)^{(1-p)/p} \tag{4.19}$$

and

$$I_2 := \left(\int_{a_{Ln}}^{\infty} \int_{a_{Ln}}^{x} |g_n' W(u)|^p \left(\frac{W(x)}{W(u)} \right)^{1/2} du\, dx \right). \tag{4.20}$$

Now by (4.8) we may write

$$\begin{aligned} I_1 &= \left(\int_{a_{Ln}}^{\infty} J_n(x)\, dx \right)^{(1-p)/p} \\ &\leqslant C \left[\sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-1)(1-p)} \times \| W(R_n^{(j)} - R^{(j)}) \|_{L_p[a_{Ln}, \infty)}^{1-p} \right. \\ &\quad \left. + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-1)(1-p)} \| WR_n^{(r)} \|_{L_p(\mathbb{R})}^{1-p} \right] \end{aligned} \tag{4.21}$$

(by Lemma 4.3).

Also

$$I_2 = \int_{a_{Ln}}^{\infty} |g_n' W(u)|^p \int_u^{\infty} \left(\frac{W(x)}{W(u)} \right)^{1/2} dx\, du.$$

Now if $x \geqslant u \geqslant a_{Ln}$, Lemma 2.1 gives

$$Q'(x) \geqslant C_1 Q'(a_{Ln}) \geqslant C_2 \frac{nT(a_n)^{1/2}}{a_n}$$

so that

$$\begin{aligned} I_2 &\leqslant C_3 \frac{a_n}{nT(a_n)^{1/2}} \int_{a_{Ln}}^{\infty} |g_n' W(u)|^p \left[W(u)^{-1/2} \int_u^{\infty} W(x)^{1/2} Q'(x)\, dx \right] du \\ &\leqslant C_4 \frac{a_n}{nT(a_n)^{1/2}} \int_{a_{Ln}}^{\infty} |g_n' W(u)|^p\, du. \end{aligned}$$

This gives

$$I_2 \leqslant C_4 \frac{a_n}{nT(a_n)^{1/2}} \| (R_n' - R') W \|_{L_p[a_{Ln}, \infty)}^{p}. \tag{4.22}$$

Combining our estimates for I_1 and I_2 gives the result. ∎

We are now in the position to give:

Proof of Theorem 4.1 *for* $0<p<1$. Let $P_n \in \mathscr{P}_n$ and $P \in \mathscr{P}_{r-1}$ be given by (4.1). We first note that if $0 \leqslant l < r$,

$$(P_n^{(l)} - P^{(l)})(a_{Ln}) = 0.$$

Thus applying (4.17) to $P_n^{(l)}$ with r in (4.17) replaced by $r-l$ gives

$$\begin{aligned} &\| W(P_n^{(l)} - P^{(l)}) \|_{L_p[a_{Ln}, \infty)} \\ &\quad \leqslant C_1 \Bigg[\Bigg[\frac{a_n}{nT(a_n)^{1/2}} \| W(P_n^{(l+1)} - P^{(l+1)}) \|^p_{L_p[a_{Ln}, \infty)} \Bigg] \\ &\quad \times \Bigg[\sum_{j=l+1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-l-1)(1-p)} \| W(P_n^{(j)} - P^{(j)}) \|^{1-p}_{L_p[a_{Ln}, \infty)} \Bigg] \\ &\quad + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-l-1)(1-p)} \| WP_n^{(r)} \|^{1-p}_{L_p(\mathbb{R})} \Bigg]. \end{aligned} \tag{4.23}$$

We show that for $k = r-1, r-2, ..., 0$

$$\| W(P_n^{(k)} - P^{(k)}) \|_{L_p[a_{Ln}, \infty)} \leqslant C_3 \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{r-k} \| WP_n^{(r)} \|_{L_p(\mathbb{R})}. \tag{4.24}$$

Firstly, if $k = r-1$, (4.23) with $l = r-1$ gives

$$\| W(P_n^{(r-1)} - P^{(r-1)}) \|_{L_p[a_{Ln}, \infty)} \leqslant C_4 \left(\frac{a_n}{nT(a_n)^{1/2}} \right) \| WP_n^{(r)} \|_{L_p(\mathbb{R})}.$$

Assume now that (4.24) holds for $r-1, ..., k+1$. We prove (4.24) for k. Substituting (4.24) with $r-1, ..., k+1$ into (4.23) with $l = k$ gives

$$\begin{aligned} &\| W(P_n^{(k)} - P^{(k)}) \|_{L_p[a_{Ln}, \infty)} \\ &\quad \leqslant C_5 \Bigg[\frac{a_n}{nT(a_n)^{1/2}} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-k-1)p} \| WP_n^{(r)} \|^p_{L_p(\mathbb{R})} \\ &\quad \times \Bigg[\sum_{j=k+1}^{r-1} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(j-k-1)(1-p)} \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-j)(1-p)} \Bigg] \\ &\quad \times \| WP_n^{(r)} \|^{1-p}_{L_p(\mathbb{R})} + \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{(r-k-1)(1-p)} \| WP_n^{(r)} \|^{1-p}_{L_p(\mathbb{R})} \Bigg] \\ &\quad \leqslant C_6 \left(\frac{a_n}{nT(a_n)^{1/2}} \right)^{r-k} \| WP_n^{(r)} \|^{1-p}_{L_p(\mathbb{R})}. \end{aligned}$$

Thus (4.24) holds for all k. In particular, we have

$$\|W(P_n - P)\|_{L_p[a_{Ln}, \infty)} \leqslant C_7 \left(\frac{a_n}{nT(a_n)^{1/2}}\right)^r \|WP_n^{(r)}\|_{L_p(\mathbb{R})}$$

$$\leqslant C_8 \left(\frac{a_n}{n}\right)^r \|WP_n^{(r)} \Phi^r_{a_n/n}(x)\|_{L_p(\mathbb{R})}. \quad \blacksquare$$

5. EQUIVALENCE OF MODULUS AND REALIZATION FUNCTIONAL

In this section we prove Theorem 1.3 which establishes the fundamental equivalence of our modulus of continuity and its corresponding realization-functional. We also deduce Corollary 1.4. Throughout for $0 < p \leqslant \infty$ we set

$$q := \min\{1, p\}.$$

We begin by quickly recalling the definitions of our moduli and realization functional. See (1.11), (1.12), and (1.17). Let $r \geqslant 1$, $0 < t \leqslant C$, and let $n = n(t)$ be determined by (1.18). Then we have

$$\text{(1)} \quad w_{r,p}(f, W, t) := \sup_{0 < h \leqslant t} \|W(\Delta^r_{h\Phi_t(x)}(f))\|_{L_p(|x| \leqslant \sigma(2t))} + \inf_{R \text{ of deg } \leqslant r-1} \|(f - R) W\|_{L_p(|x| \geqslant \sigma(4t))} \tag{5.1}$$

$$\text{(2)} \quad \bar{w}_{r,p}(f, W, t) := \left[\frac{1}{t}\int_0^t \|W(\Delta^r_{h\Phi_t(x)}(f))\|^p_{L_p(|x| \leqslant \sigma(2t))}\, dh\right]^{1/p} + \inf_{R \text{ of deg } \leqslant r-1} \|(f - R) W\|_{L_p(|x| \geqslant \sigma(4t))}, \tag{5.2}$$

where we set $\bar{w} = w$ for $p = \infty$ and

$$\text{(3)} \quad K_{r,p}(f, W, t^r) := \inf_{P \in \mathscr{P}_n} \{\|(f - P) W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)} \Phi^r_t(x) W\|_{L_p(\mathbb{R})}\}. \tag{5.3}$$

We begin with our lower bound.

Lemma 5.1. *Let $W \in \mathscr{E}_1$, assume (1.20), and let $L > 0$ be fixed. Let $r \geqslant 1$, $0 < p \leqslant \infty$, and $0 < t < C$. Then there exists $C_1 \neq C_1(f, t)$ such that*

$$w_{r,p}(f, W, Lt) \leqslant C_1 K_{r,p}(f, W, t^r). \tag{5.4}$$

Proof. Let $q = \min\{1, p\}$. Then by (2.12), there exists u such that $4Lt = a_u/u$. Now let $n = n(t)$ be determined by (1.18) and recall it has the form

$$n = \inf\left\{k: \frac{a_k}{k} \leqslant t\right\}.$$

Thus by (2.25) and (2.18) we have

$$\frac{a_n}{2n} \leqslant \frac{t}{2} < \frac{a_n}{n} \tag{5.5a}$$

and

$$\Phi_t(x) \sim \Phi_{a_n/n}(x) \sim \Phi_{Lt}(x) \qquad \forall x \in \mathbb{R}, \tag{5.5b}$$

where the constants in the $\sim$ relation are independent of t and x. Also by (2.13) and (2.26), $\exists \beta > 0$ such that

$$\sigma(4Lt) = \sigma\left(\frac{a_u}{u}\right) \geqslant a_{u/2} \geqslant a_{\beta n}. \tag{5.6}$$

Choose $P \in \mathscr{P}_n$ such that

$$\|(f - P)\, W\|_{L_p(\mathbb{R})} + t^r\, \|P^{(r)} \Phi_t^r\, W\|_{L_p(\mathbb{R})} \leqslant 2K_{r,\, p}(f, W, t^r). \tag{5.7}$$

We show that

$$\sup_{0 < h \leqslant tL} \|W(\Delta^r_{h\Phi_{tL}(x)}(f))\|_{L_p(|x| \leqslant \sigma(2Lt))} \leqslant C_6 K_{r,\, p}(f, W, t^r) \tag{5.8}$$

and

$$\inf_{R \text{ of deg } \leqslant r-1} \|(f - R)\, W\|_{L_p(|x| \geqslant \sigma(4Lt))} \leqslant C_2 K_{r,\, p}(f, W, t^r). \tag{5.9}$$

This then gives (5.4) using the definition (5.1).

We begin with:

Proof of (5.9). We appeal to Theorem 4.1 and choose for our given $P, S \in \mathscr{P}_{r-1}$ as in (4.1), so that (4.2) holds. Next we recall Lemma 3.1 from [8]. Let W be an even weight. Then for f satisfying $fW \in L_p(\mathbb{R})$ and for $\xi > 0$,

$$\inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\, W\|_{L_p(|x| \geqslant \xi)} \leqslant 2^{4/q-3} \left[\inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\, W\|_{L_p(x \geqslant \xi)} \right.$$

$$\left. + \inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\, W\|_{L_p(x \leqslant -\xi)} \right].$$

We apply the above with $\xi := \sigma(4Lt)$. In particular, we estimate

$$\inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\, W\|^q_{L_p(x \geqslant \sigma(4Lt))}.$$

The other term can be handled similarly. Thus

$$\begin{aligned}
&\inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\, W\|^q_{L_p(x \geqslant \sigma(4Lt))} \\
&\quad \leqslant \|(f-S)\, W\|^q_{L_p(x \geqslant \sigma(4Lt))} \\
&\quad \leqslant \|(f-P)\, W\|^q_{L_p(x \geqslant \sigma(4Lt))} + \|(P-S)\, W\|^q_{L_p(x \geqslant \sigma(4Lt))} \\
&\quad \leqslant C_3(K_{r,\,p}(f, W, t^r))^q + \|(P-S)\, W\|^q_{L_p(x \geqslant a_{\beta n})} \qquad \text{(by (5.6) and (5.7))} \\
&\quad \leqslant C_3(K_{r,\,p}(f, W, t^r))^q + C_4 t^r \, \|P^{(r)} \Phi_t^r\, W\|^q_{L_p(\mathbb{R})} \\
&\qquad\qquad \text{(by (4.2), (5.5a), and (5.5b))} \\
&\quad \leqslant C_5(K_{r,\,p}(f, W, t^r))^q.
\end{aligned}$$

Hence (5.9).

Next we proceed with:

Proof of (5.8). Let $0 < h \leqslant Lt$ and write

$$\begin{aligned}
\|\, W(\Delta^r_{h\Phi_{tL}(x)}(f))\|^q_{L_p(|x| \leqslant \sigma(2Lt))} &\leqslant \|\, W(\Delta^r_{h\Phi_{tL}(x)}(f-P))\|^q_{L_p(|x| \leqslant \sigma(2Lt))} \\
&\quad + \|\, W(\Delta^r_{h\Phi_{Lt}(x)}(P))\|^q_{L_p(|x| \leqslant \sigma(2Lt))} \\
&= I_1 + I_2.
\end{aligned}$$

We first deal with the estimation of I_1. Note that given $A > 0$,

$$|x| \leqslant \sigma(2Lt)$$

implies

$$1 - \frac{|x|}{\sigma(tL)} \geqslant 1 - \frac{\sigma(2Lt)}{\sigma(tL)} \geqslant \frac{C_7}{T(\sigma(Lt))} \geqslant \left(\frac{At}{\sigma(tL)}\right)^2$$

by (2.14) and (2.15) provided t is small enough. Thus (2.31) and (2.35) are satisfied so that by (2.36),

$$I_1 \leqslant C_6 \, \|W(f-P)\|^q_{L_p(\mathbb{R})} \leqslant C_7 K_{r,\,p}(f,\, W,\, t^r)^q \tag{5.10}$$

by (5.7).

To deal with the estimation of I_2 we observe first much as in [8] that for

$$S(w) := \sum_{l=0}^{r-1} \frac{P^{(l)}(x)(w-x)^l}{l!} \in \mathscr{P}_{r-1}$$

we have by (2.28) that $\Delta^r_{h\Phi_{Lt}(x)} S \equiv 0$.

Thus expanding $P(x+((r/2)-k)h\Phi_t(x))$, $0 \leqslant k \leqslant r$, in a power series about x gives

$$\begin{aligned}
\Delta^r_{h\Phi_{tL}(x)} P(x) &= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \, P\left(x+\left(\frac{r}{2}-k\right) h\Phi_{tL}(x)\right) \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left[\sum_{l=0}^{r-1} + \sum_{l=r}^{n}\right] \frac{[((r/2)-k)\, h\Phi_{tL}(x)]^l \, P^{(l)}(x)}{l!} \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{[((r/2)-k)\, h\Phi_{tL}(x)]^l \, P^{(l)}(x)}{l!},
\end{aligned}$$

so that

$$\begin{aligned}
I_2 &\leqslant C_8 \sum_{k=0}^{r} \binom{r}{k^q} \sum_{l=r}^{n} \left[\frac{((r/2)h)^{lq}}{l!^q}\right] \|P^{(l)}\Phi^l_{tL} W\|^q_{L_p(|x| \leqslant \sigma(2Lt))} \\
&\leqslant C_9 2^{rq} h^{rq} \sum_{l=r}^{n} \left[\frac{((r/2)h)^{(l-r)q}}{l!^q}\right] \|P^{(l)}\Phi^l_{tL} W\|^q_{L_p(|x| \leqslant \sigma(2Lt))}.
\end{aligned} \tag{5.11}$$

Now by repeated applications of Theorem 3.1, we have by using (5.5),

$$\begin{aligned}
&\|P^{(l)}\Phi^l_{tL} W\|_{L_p(\mathbb{R})} \\
&\qquad \leqslant C^r_{10} \, \|P^{(r)}\Phi^r_{tL} W\|_{L_p(\mathbb{R})} \, C^{l-r}_{11} \prod_{j=r}^{l-1} \left(\frac{n}{a_n} + \frac{j}{a_n} \, T(a_n)^{1/2}\right),
\end{aligned} \tag{5.12}$$

where C_j, $j = 10, 11$ are independent of n, x, l, L, and h. Now we observe using (2.6) that given $\varepsilon > 0$, we have for n large enough and $r \leqslant l \leqslant n$

$$\prod_{j=r}^{l-1} \left(\frac{n}{a_n} + \frac{j}{a_n} \, T(a_n)^{1/2}\right) \leqslant C_{12} \varepsilon^{l-r} \left(\frac{n}{a_n}\right)^{l-r} l!. \tag{5.13}$$

Here it is important that C_{12} does not depend on l, n, h, or L and that C_{10} and C_{11} above are independent of ε.

We may now substitute (5.13) into (5.12) so that (5.11) becomes

$$I_2 \leqslant C_{13} h^{rq} \, \|P^{(r)} \Phi_t^r W\|_{L_p(\mathbb{R})}^q \sum_{l=r}^{n} \left[\frac{((r/2)\, h C_{10} C_{11} \varepsilon(n/a_n))^{(l-r)q} \, l!^q}{l!^q} \right]$$

$$\leqslant C_{14} t^{rq} \, \|P^{(r)} \Phi_t^r W\|^q \sum_{k=0}^{\infty} \left[\frac{1}{2} \right]^k \qquad \text{(if } \varepsilon \text{ is small enough)},$$

$$\leqslant C_{15} t^{rq} \, \|P^{(r)} \Phi_t^r W\|^q \leqslant C_{16} K_{r,p}(f, W, t^r)^q. \tag{5.14}$$

Thus combining (5.10) and (5.14) and taking sup s over $0 \leqslant h \leqslant Lt$ gives (5.8). ∎

We proceed with the upper bound. This is more difficult than the lower bound and does not follow as easily using, for example, the methods of [8]. The crux is establishing the following quasi monotonicity type property of $\bar{w}$.

LEMMA 5.2. *There exist* C_j, $j = 1, 2$, *and* $0 < \varepsilon_0 < 1$ *such that if* $0 < \lambda < \varepsilon_0$ *and* $0 < s, t < C_1$ *with*

$$\lambda \leqslant \frac{s}{t} \leqslant \varepsilon_0 \tag{5.15}$$

we have

$$\bar{w}_{r,p}(f, W, s) \leqslant C_2 \bar{w}_{r,p}(f, W, t). \tag{5.16}$$

It is important that the C_j, $j = 1, 2$, and ε_0 do not depend on f, s, and t but depend on λ.

Remark. We remark that the above property is by no means obvious as we recall that our modulus is not necessarily monotone increasing. We prove it for $p < \infty$ as the case $p = \infty$ is much easier.

Proof. Let us write

$$\begin{aligned}\bar{w}_{r,p}(f, W, s)^p \leqslant \frac{2^p}{s} \Bigg[\int_0^s & \|W(\Delta^r_{h\Phi_s(x)}(f))\|^p_{L_p(|x| \leqslant \sigma(3t))} \\ & + \|W(\Delta^r_{h\Phi_s(x)}(f))\|^p_{L_p(\sigma(3t) \leqslant |x| \leqslant \sigma(2s))} \, dh \Bigg] \\ & + 2^p \inf_{R \text{ of deg } \leqslant r-1} \|(f - R) W\|^p_{L_p(|x| \geqslant \sigma(4s))} \\ & = I_1 + I_2. \end{aligned} \tag{5.17}$$

Firstly, by choice of s and t, $s/t \leqslant 1$ so that

$$\sigma(4s) \geqslant \sigma(4t)$$

(recall σ is decreasing). Thus

$$\begin{aligned} I_2 &\leqslant 2^p \inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\,W\|^p_{L_p(|x| \geqslant \sigma(4t))} \\ &\leqslant 2^p \bar{w}^p_{r,\,p}(f, W, t). \end{aligned} \tag{5.18}$$

Next we estimate I_1.

Write $I_1 \leqslant I_3 + I_4$, where

$$I_3 := \frac{2^p}{s} \int_0^s \|W(\Delta^r_{h\Phi_s(x)}(f))\|^p_{L_p(|x| \leqslant \sigma(3t))}\, dh$$

and

$$I_4 := \frac{2^p}{s} \int_0^s \|W(\Delta^r_{h\Phi_s(x)}(f))\|^p_{L_p(\sigma(3t) \leqslant |x| \leqslant \sigma(2s))}\, dh.$$

We begin with the estimation of I_4. To this end we make use of Lemma 2.6. Much as in the proof of Lemma 5.1, we have

$$I_4 \leqslant C_1 \inf_{R \text{ of deg } \leqslant r-1} \|(f-R)\,W\|^p_{L_p(|x| \geqslant \sigma(4t))} \leqslant C_1 \bar{w}_{r,\,p}(f, W, t)^p. \tag{5.19}$$

Here we used that for small t,

$$\begin{aligned} \inf\{x - Mrs\Phi_s(x) \colon \sigma(3t) \leqslant x \leqslant \sigma(2s)\} &= \sigma(3t) - Mrs\Phi_s(\sigma(3t)) \\ &\geqslant \sigma(3t) - Ct\Phi_t(\sigma(3t)) \\ &\geqslant \sigma(3t) - CtT(\sigma(t))^{-1/2} \\ &\geqslant \sigma(3t) + o(1/T(\sigma(t))) \geqslant \sigma(4t) \end{aligned}$$

by (2.3), (2.6), (2.14), (2.18), and as Φ_s is decreasing in $[0, \sigma(2s)]$.

It remains to estimate I_3.

As s and t are small enough, we can use (2.12), (2.19), and (2.30) to obtain a large enough positive integer n such that $a_n/n \sim s$ and then divide $J := [-\sigma(3t), \sigma(3t)]$ into $O(1/s)$ intervals J_k such that

$$|J_k| \leqslant s\Phi_s(x), \qquad x \in J_k.$$

Formally, we do this by choosing a partition

$$-\sigma(3t) = \tau_0 < \tau_1 \cdots < \tau_n = \sigma(3t)$$

with

$$\frac{\int_{\tau_k}^{\tau_{k+1}} \Phi_s^{-1}(x)\,dx}{\int_{\tau_0}^{\tau_n} \Phi_s^{-1}(x)\,dx} = \frac{1}{n}, \qquad 0 \leqslant k \leqslant n$$

and set

$$J_k = [\tau_k, \tau_{k+1}].$$

Then if $|J_k|$ denotes the length of J_k we have

$$\text{(1)} \qquad \Phi_s(x) \sim \Phi_s(y), \qquad x, y \in J_k \tag{5.20}$$

and

$$\text{(2)} \qquad W(x) \sim W(y), \qquad x, y \in J_k.$$

Here the constants in the $\sim$ relation are independent of x, y, s, k.

Then

$$\begin{aligned} I_3 &= \frac{2^p}{s} \int_0^s \| W(\Delta^r_{h\Phi_s(x)}(f)) \|^p_{L_p(|x| \leqslant \sigma(3t))}\, dh \\ &\leqslant C_2 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^s |\Delta^r_{h\Phi_s(x)}(f)|^p\, dh\, dx \\ &= C_2 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^{s\Phi_s(x)/\Phi_t(x)} |\Delta^r_{u\Phi_t(x)}(f)|^p \frac{\Phi_t(x)}{\Phi_s(x)}\, du\, dx. \end{aligned}$$

Now we may rewrite (2.17) for the given s and t as

$$\sup_{x \in \mathbb{R}} \frac{s\Phi_s(x)}{t\Phi_t(x)} \leqslant C \frac{s}{t} \sqrt{\log\left(2 + \frac{t}{s}\right)}$$

for some $C \neq C(s, t)$. It follows that

$$\sup_{x \in \mathbb{R}} \frac{s\Phi_s(x)}{t\Phi_t(x)} \leqslant 1$$

if

$$s/t \leqslant \varepsilon_0,$$

where ε_0 is independent of s and t. Then if $\lambda < \varepsilon_0$, we have for $\lambda \leqslant s/t \leqslant \varepsilon_0$,

$$C_3 \leqslant \frac{\Phi_s(x)}{\Phi_t(x)} \leqslant C_4 \qquad \forall x \in \mathbb{R},$$

where C_3 and C_4 are independent of s, t, and ε_0. Then

$$\begin{aligned} I_3 &\leqslant C_5 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^t |\Delta^r_{u\Phi_t(x)}(f)|^p \, du \, dx \\ &\leqslant C_6 \frac{1}{t} \int_0^t \| W(\Delta^r_{h\Phi_t(x)}(f)) \|^p_{L_p(|x| \leqslant \sigma(2t))} \, dh \\ &\leqslant C_6 \bar{w}_{r,p}(f, W, t)^p. \end{aligned} \tag{5.21}$$

Combining our estimates (5.18), (5.19), and (5.21) gives the lemma. ∎

LEMMA 5.3. *Let* $W \in \mathscr{E}_1$ *and assume* (1.20). *Let* $r \geqslant 1$ *and* $0 < p \leqslant \infty$. *Then for* $0 < t < C_1$, *there exists* C_2 *and* C_3 *independent of* f *and* t *such that*

$$K_{r,p}(f, W, t^r) \leqslant C_2 \bar{w}_{r,p}(f, W, C_3 t). \tag{5.22}$$

Proof. Put $t/2 = a_u/u$ for some $u \geqslant u_0$ and let $n = n(t)$ be determined by (1.18), so that

$$n = \inf \left\{ k \colon \frac{a_k}{k} \leqslant \frac{2a_u}{u} \right\}$$

and

$$\frac{1}{2} \frac{a_n}{n} \leqslant \frac{a_u}{u} < \frac{a_n}{n}. \tag{5.23}$$

Now it is easy to see that for large enough u and the given n,

$$t = 2 \frac{a_u}{u} = \frac{a_n}{n} \lambda(n) C$$

for some $\lambda(n) \in [\frac{4}{5}, 1]$ and $C > 0$ independent of n. We then apply (1.14), and choose $P \in \mathscr{P}_n$ such that

$$\|(f - P) W\|_{L_p(\mathbb{R})} \leqslant C_1 \bar{w}_{r,p}(f, W, C_2 t) \tag{5.24}$$

for some C_1 and C_2 independent of f and t.

We show that for some $C_3 \neq C_3(f, t)$,

$$t^r \, \|P^{(r)} \Phi^r_t W\|_{L_p(\mathbb{R})} \leqslant C_3 \bar{w}_{r,p}(f, W, C_2 t) \tag{5.25}$$

for then by (5.24),

$$K_{r,p}(f, W, t^r) = \inf_{R \in \mathscr{P}_n} \{\|(f-R)W\|_{L_p(\mathbb{R})} + t^r \|R^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}\}$$
$$\leqslant \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}$$
$$\leqslant (C_1 + C_3)\,\bar{w}_{r,p}(f, W, C_2 t).$$

Thus we show (5.25).

Now let $\delta > 0$ be a small enough positive number and put $s := \delta t$. It is sufficient at this point of the proof to choose δ small enough so that by Lemma 5.2,

$$\bar{w}_{r,p}(f, W, s) \leqslant C_4 \bar{w}_{r,p}(f, W, C_2 t). \tag{5.26}$$

Later, we will need to choose δ smaller still.

Let us recall much as in Lemma 5.1 that we have for $0 < h \leqslant s$

$$\Delta^r_{h\Phi_s(x)} P(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{[((r/2)-k)\, h\Phi_s(x)]^l\, P^{(l)}(x)}{l!}. \tag{5.27}$$

Applying (5.27) to $x^r \in P_r$ and using (2.28) gives

$$(r!)^{-1}\, \Delta^r_{h\Phi_s(x)} x^r = (h\Phi_s(x))^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{[((r/2)-k)\, h\Phi_s(x)]^r}{r!}. \tag{5.28}$$

We now combine (5.27) and (5.28) together with (3.5) to give much as in (5.14),

$$\|W\Delta^r_{h\Phi_s(x)} P(x) - W(h\Phi_s(x))^r P^{(r)}(x)\|^q_{L_p(|x| \leqslant \sigma(2s))}$$
$$\leqslant C_5 h^{rq}\, \|WP^{(r)}\Phi^r_s(x)\|^q_{L_p(\mathbb{R})} \sum_{l=r+1}^{n} \frac{(C_6 (n/a_n)\, h)^{(l-r)q}\, l!^q}{l!^q}, \tag{5.29}$$

where C_6 is independent of t, n, h, P_n, and l.

Now by (2.26), (2.4), and (5.23) we can choose $\alpha > 3$ independent of t, n, h, P_n, l, and C_2 such that $a_u < a_{\alpha n}$. Further (if necessary) we make δ in the definition of s smaller still so that

$$\delta < \min\left(\frac{1}{8\alpha}, \frac{1}{2}\right) \tag{5.30}$$

and

$$2s \leqslant \frac{t}{4\alpha} \leqslant \frac{a_{\alpha n}}{\alpha n}.$$

This gives

$$\sigma(2s) \geqslant \sigma\left(\frac{t}{4\alpha}\right) \geqslant \sigma\left(\frac{a_{\alpha n}}{\alpha n}\right) \geqslant a_{\xi n} \tag{5.31}$$

for some fixed $3 < \xi < \alpha$.

It follows that we obtain using (5.31), (2.18), and (3.12),

$$\begin{aligned} &\| W\Delta^r_{h\Phi_s(x)} P(x) - W(h\Phi_s(x))^r P^{(r)}(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \\ &\quad \leqslant \tfrac{1}{2} h^{rq} \| WP^{(r)}\Phi^r_s(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \end{aligned} \tag{5.32}$$

provided $(n/a_n)\, h) \leqslant \Delta$, where Δ is a fixed positive small number independent of t, h, n, P_n, and l.

Now by (5.30) and (5.23), it is easy to see that $\Delta s \leqslant \Delta(a_n/n)$ so that $\forall 0 < h \leqslant \Delta s$ we have

$$\begin{aligned} &\| W\Delta^r_{h\Phi_s(x)} P(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \\ &\quad \geqslant h^{rq} \| W(\Phi_s(x))^r P^{(r)}(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \\ &\qquad - \| W\Delta^r_{h\Phi_s(x)} P(x) - W(h\Phi_s(x))^r P^{(r)}(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \\ &\quad \geqslant \tfrac{1}{2} h^{rq} \| WP^{(r)}\Phi^r_s(x) \|^q_{L_p(|x| \leqslant \sigma(2s))} \qquad \text{(by (5.32))} \\ &\quad \geqslant C_7 h^{rq} \| WP^{(r)}\Phi^r_s(x) \|^q_{L_p(\mathbb{R})} \end{aligned} \tag{5.33}$$

by (3.12). Now raising (5.33) to the p/qth powers, integrating for h from 0 to Δs using the fact that $\Phi_s(x) \sim \Phi_t(x)$, $x \in \mathbb{R}$ (see (2.18)), and assuming that $\Delta < 1$ as we may, gives

$$\begin{aligned} t^{rp} \| WP^{(r)}\Phi^r_t(x) \|^p_{L_p(\mathbb{R})} &\leqslant \frac{C_8}{s} \int_0^{\Delta s} \| W\Delta^r_{h\Phi_s(x)} P(x) \|^p_{L_p(|x| \leqslant \sigma(2s))}\, dh \\ &\leqslant \frac{C_8}{s} \int_0^{s} \| W\Delta^r_{h\Phi_s(x)} P(x) \|^p_{L_p(|x| \leqslant \sigma(2s))}\, dh \\ &\leqslant \frac{C_8}{s} \int_0^{s} \| W\Delta^r_{h\Phi_s(x)} (P-f)(x) \|^p_{L_p(|x| \leqslant \sigma(2s))}\, dh \\ &\quad + \frac{C_8}{s} \int_0^{s} \| W\Delta^r_{h\Phi_s(x)} f(x) \|^p_{L_p(|x| \leqslant \sigma(2s))}\, dh \\ &\leqslant C_9 \{ \| W(P-f) \|^p_{L_p(\mathbb{R})} + \bar{w}_{r,p}(f, W, s) \} \qquad \text{(by (2.36))} \\ &\leqslant C_{10} \bar{w}_{r,p}(f, W, C_2 t) \end{aligned}$$

by (5.26) and (5.24). Thus we have (5.25) and the lemma. ∎

We now combine Lemmas 5.1 and 5.3 to give

Proof of Theorem 1.3. We have for any $L>0$ and $0<t<t_0$,

$$\begin{aligned}\bar{w}_{r,p}(f, W, Lt) &\leqslant w_{r,p}(f, W, Lt) \leqslant C_1 K_{r,p}(f, W, t^r)\\ &\leqslant C_2 \bar{w}_{r,p}(f, W, C_3 t) \leqslant C_2 w_{r,p}(f, W, C_3 t),\end{aligned} \tag{5.34}$$

where C_3 is independent of L, f, and t while C_1 and C_2 are independent of f and t but depend on L.

Fix $M>0$ and choose $L=MC_3$ and $s=C_3 t$ to deduce that

$$w_{r,p}(f, W, Ms) \leqslant C_2 w_{r,p}(f, W, s) \tag{5.35}$$

and so we have the upper bound in (1.23). Similarly (5.34) gives

$$\bar{w}_{r,p}(f, W, Ms) \leqslant C_2 \bar{w}_{r,p}(f, W, s). \tag{5.36}$$

Then (5.34) gives

$$w_{r,p}(f, W, s) \sim \bar{w}_{r,p}(f, W, s) \sim K_{r,p}(f, W, s^r)$$

with constants independent of f and s. The proof of the lower bound of (1.23) is similar and easier. ∎

6. THE PROOFS OF THEOREM 1.5 AND COROLLARIES 1.6 AND 1.7

We begin with:

Proof of Theorem 1.5. For each $n \geqslant 0$, choose P_n^* to be the best approximant to f satisfying

$$\|(f-P_n^*) W\|_{L_p(\mathbb{R})} = E_n[f]_{W,p}.$$

Here, we set $P_{2^{-1}}^* = P_0^*$. Now let $t>0$ be small enough and define n by (1.18). Put $l=[\log_2 n]=$ the largest integer $\leqslant \log_2 n$ so that $2^l \leqslant n < 2^{l+1}$.

Then by Theorem 1.3 and Corollary 1.4

$$\begin{aligned} w_{r,p}\left(f, W, \frac{a_n}{n}\right)^q &\\ &\leqslant C_1 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q\\ &\leqslant C_2 \left[\, \|(f-P_{2^l}^*) W\|_{L_p(\mathbb{R})}^q + \left(\frac{a_n}{n}\right)^{rq} \|P_{2^l}^{*(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})}^q \right]\end{aligned}$$

$$\leqslant C_3\left[E_{2^l}[f]^q_{W,p}+\left(\frac{a_n}{n}\right)^{rq}\sum_{k=-1}^{l-1}\|[P^*_{2^{k+1}}-P^*_{2^k}]^{(r)}\Phi^r_{a_n/n}W\|^q_{L_p(\mathbb{R})}\right]$$

$$\leqslant C_4\left[E_{2^l}[f]^q_{W,p}\right.$$
$$\left.+\left(\frac{a_n}{n}\right)^{rq}\sum_{k=-1}^{l-1}\left\|[P^*_{2^{k+1}}-P^*_{2^k}]^{(r)}\Phi^r_{a_{2^{k+1}}/2^{k+1}}(\log(2^{l-k}))^{r/2}W\right\|^q_{L_p(\mathbb{R})}\right]$$

as $r\geqslant 1$ and by (2.17). This can be continued as

$$\leqslant C_5\left[E_{2^l}[f]^q_{W,p}\right.$$
$$\left.+\left(\frac{a_n}{n}\right)^{rq}\sum_{k=-1}^{l-1}(l-k+1)^{rq/2}\left(\frac{2^k}{a_{2^k}}\right)^{rq}\|[P^*_{2^{k+1}}-P^*_{2^k}]W\|^q_{L_p(\mathbb{R})}\right]$$

by (1.20).

We can continue this as

$$\leqslant C_6\left[E_{2^l}[f]^q_{W,p}+\left(\frac{a_n}{n}\right)^{rq}\sum_{k=-1}^{l-1}(l-k+1)^{rq/2}\left(\frac{2^k}{a_{2^k}}\right)^{rq}E_{2^k}[f]^q_{W,p}\right]$$

$$\leqslant C_7\left(\frac{a_n}{n}\right)^{rq}\left[\sum_{k=-1}^{l}(l-k+1)^{rq/2}\left(\frac{2^k}{a_{2^k}}\right)^{rq}E_{2^k}[f]^q_{W,p}\right]. \tag{6.1}$$

Now by (2.25) we have that $t\sim a_n/n$. Also by (2.18),

$$\Phi_t(x)\sim\Phi_{a_n/n}(x),\qquad x\in\mathbb{R}$$

so that by Theorem 1.3

$$K_{r,p}(f,W,t^r)\sim K_{r,p}\left(f,W,\left(\frac{a_n}{n}\right)^r\right)$$

and

$$w_{r,p}(f,W,t)\sim w_{r,p}\left(f,W,\frac{a_n}{n}\right). \tag{6.2}$$

Thus (6.2) becomes

$$w_{r,p}(f,W,t)^q\leqslant C_8t^{rq}\left[\sum_{k=-1}^{l}(l-k+1)^{rq/2}\left(\frac{2^k}{a_{2^k}}\right)^{rq}E_{2^k}[f]^q_{W,p}\right],$$

where $C_8\neq C_8(f,t)$. ∎

We deduce

Proof of Corollary 1.6. Suppose first that

$$w_{r,p}(f, W, t) = O(t^{\alpha}), \qquad t \to 0^+$$

Then in particular

$$w_{r,p}\left(f, W, \frac{a_n}{n}\right) = O\left(\left(\frac{a_n}{n}\right)^{\alpha}\right), \qquad n \to \infty,$$

so that by Corollary 1.4

$$E_n[f]_{W,p} = O\left(\left(\frac{a_n}{n}\right)^{\alpha}\right).$$

Next suppose $E_n[f]_{W,p} = O((a_n/n)^{\alpha})$. Let $0 < \varepsilon < 1$. Then, by (1.25)

$$\begin{aligned}
w_{r,p}\left(f, W, \frac{a_n}{n}\right) &\leqslant C_1 \left(\frac{a_n}{n}\right)^{r} \left[\sum_{k=-1}^{l} (l-k+1)^{rq/2} \left(\frac{2^k}{a_{2^k}}\right)^{(r-\alpha)q}\right]^{1/q} \\
&\leqslant C_1 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{k=-1}^{l} (l-k+1)^{rq/2} \left(\frac{a_n/n}{a_{2^k}/2^k}\right)^{(r-\alpha)q}\right]^{1/q} \\
&\leqslant C_2 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{k=-1}^{l} (l-k+1)^{rq/2} \left(\frac{2^{l+1}}{2^k}\right)^{(r-\alpha)q(-1+\varepsilon)}\right]^{1/q} \\
&\qquad \text{(by (2.11))} \\
&\leqslant C_3 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{j=0}^{\infty} j^{(r/2)q} a^{jq}\right]^{1/q} \qquad \text{(for some } 0 < a < 1) \\
&\leqslant C_4 \left(\frac{a_n}{n}\right)^{\alpha}.
\end{aligned} \tag{6.3}$$

Now for $t > 0$ small enough, we may determine n by (1.18) and using Theorem 1.3, (2.25), and (6.2) deduce the corollary for t. ∎

We now proceed to prove Corollary 1.7. We need first a lemma that will prove useful in other related contexts.

LEMMA 6.1. *Let* $W \in \mathscr{E}_1$, $r \geqslant 1$, $0 < p \leqslant \infty$, *and assume* (1.20). *Then for* $n \geqslant C$ *and* $\forall P_n \in \mathscr{P}_n$ *satisfying*

$$\|(f - P_n) W\|_{L_p(\mathbb{R})} \leqslant L E_n[f]_{W,p} \tag{6.4}$$

for some $L \geqslant 1$, *we have*

$$\|(f-P_n)W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \|P_n \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})} \sim K\left(f, W, \left(\frac{a_n}{n}\right)^r\right), \tag{6.5}$$

where the constants in the $\sim$ *relation depend on L but are independent of n and f.*

We remark that in particular, (6.4) holds for P_n^* the best approximant to f.

Proof. Let $P_n^{\#}$ satisfy the required hypotheses. Then by the definition of $K_{r,p}(f, W, (a_n/n)^r)$, we have

$$\left\{\|(f-P_n^{\#})W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \|P_n^{\#(r)} \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})}\right\} \geqslant K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right). \tag{6.6}$$

Next choose P_n such that

$$\left\{\|(f-P_n)W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \|P_n^{(r)} \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})}\right\} \leqslant 2K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right). \tag{6.7}$$

Then

$$\begin{aligned}\|(P_n-P_n^{\#})W\|^q_{L_p(\mathbb{R})} &\leqslant \|(P_n-f)W\|^q_{L_p(\mathbb{R})} + \|(f-P_n^{\#})W\|^q_{L_p(\mathbb{R})} \\ &\leqslant C_1 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q \end{aligned} \tag{6.8}$$

by (6.7).

Further using (1.20), we can write using (6.8)

$$\begin{aligned}\|(P_n-P_n^{\#})^{(r)} \Phi^r_{a_n/n} W\|^q_{L_p(\mathbb{R})} &\leqslant C_2 \left(\frac{n}{a_n}\right)^{rq} \|(P_n-P_n^{\#})W\|^q_{L_p(\mathbb{R})} \\ &\leqslant C_3 \left(\frac{n}{a_n}\right)^{rq} K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q. \end{aligned} \tag{6.9}$$

Thus by (6.8) and (6.9)

$$\left(\frac{a_n}{n}\right)^{rq} \|P_n^{\#(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})}^q$$

$$\leqslant C_4 \left[\left(\frac{a_n}{n}\right)^{rq} \|P_n^{(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})}^q + \left(\frac{a_n}{n}\right)^{rq} \|(P_n - P_n^{\#})^{(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})}^q \right]$$

$$\leqslant C_5 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q, \tag{6.10}$$

so that (6.6) and (6.10) give the result. ▮

We can now give:

Proof of Corollary 1.7(a). We shall show that

$$\|W\Delta_{h\Phi_t(x)}^r(f, x, \mathbb{R})\|_{L_p[|x| \leqslant \sigma(2t)]} \leqslant C_1 t^r \|f^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})} \tag{6.11}$$

and

$$\inf_{P \in \mathscr{P}_{r-1}} \|W(f-P)\|_{L_p[|x| \geqslant \sigma(4t)]} \leqslant C_2 t^r \|f^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}. \tag{6.12}$$

We begin with:

Proof of (6.11). We begin with an observation.
If $h > 0$ we may write

$$|\Delta_h^r(f, x, \mathbb{R})| = \left| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f^{(r)}(x + t_1 + \cdots + t_r)\, dt_1\, dt_2 \cdots dt_r \right|$$

$$\leqslant h^{r-1} \int_{-hr/2}^{hr/2} |f^{(r)}(x+s)|\, ds. \tag{6.13}$$

Now note that for $s \in [-rh\Phi_t(x)/2, rh\Phi_t(x)/2]$ and $x \in [-\sigma(2t), \sigma(2t)]$ we have by (2.26)

$$\Phi_t(x) \sim \Phi_t(x+s).$$

Thus we may deduce from (6.13) that for $|x| \leqslant \sigma(2t)$ as

$$|W\Delta_{h\Phi_t(x)}^r(f, x, \mathbb{R})|$$

$$\leqslant C_3 h^r \frac{1}{(rh\Phi_t(x)/2)} \int_{-rh\Phi_t(x)/2}^{rh\Phi_t(x)/2} |Wf^{(r)}\Phi_t^r(x+s)|\, ds. \tag{6.14}$$

Case 1. $p>1$. We recall the definition of the maximal function operator

$$M[gt](x):=\sup_{u>0}\frac{1}{2u}\int_{-u}^{u}|g(x+s)|\,ds$$

which is bounded from L_p to L_p, $1<p\leqslant\infty$. It follows that (6.14) can be rewritten as

$$\begin{aligned}\|W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\|_{L_p[|x|\leqslant\sigma(2t)]}&\leqslant C_4h^r\,\|M[W\Phi_t^r f^{(r)}]\|_{L_p(\mathbb{R})}\\&\leqslant C_5t^r\,\|f^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}.\end{aligned}$$

Case 2. $p=1$. Integrating (6.14) and noting that if $u=x+s$, then for the range of x and s above,

$$\Phi_t(x)\sim\Phi_t(x+s),$$

we obtain

$$\begin{aligned}&\int_{|x|\leqslant\sigma(2t)}|W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})|\,dx\\&\quad\leqslant C_6h^{r-1}\int_{|x|\leqslant\sigma(2t)}\frac{1}{\Phi_t(x)}\int_{|s|\leqslant(rh/2)\,\Phi_t(x)}|Wf^{(r)}\Phi_t^r|\,(x+s)\,ds\,dx\\&\quad\leqslant C_7h^{r-1}\int_{\substack{u:=x+s:|x|\leqslant\sigma(2t)\\|s|\leqslant(r/2)h\Phi_t(x)}}\frac{1}{\Phi_t(u)}|Wf^{(r)}\Phi_t^r|\,(u)\int_{|s|\leqslant(rh/2)\,\Phi_t(u)}ds\,du\\&\quad\leqslant C_8h^r\int_{\mathbb{R}}|f^{(r)}W\Phi_t^r|\,(u)\,du.\end{aligned}$$

Next we give:

Proof of (6.12). We mimic the proof of (4.2) for $p>1$. For the given $t>0$, write $4t=a_u/u$. Determine $n=n(t)$ by (1.18) and recall $u\sim n$ (see (2.26)) so that

$$\begin{aligned}&\text{(a)}\quad \sigma(4t)\leqslant a_u\leqslant a_{\alpha n}\\&\text{(b)}\quad \sigma(4t)\geqslant a_{u/2}\geqslant a_{\beta n}\end{aligned}\tag{6.15}$$

for some $\alpha>1$ and $\beta>0$.

As in the proof of (5.9), we may without loss of generality suppose that $x>0$. Suppose first that $r=1$. We have

$$\begin{aligned}
\inf_{P \in \mathscr{P}_{r-1}} \|W(f-P)\|_{L_p[x \geqslant \sigma(4t)]} &\leqslant \|W(f-f(a_{\beta n}))\|_{L_p[x \geqslant a_{\beta n}]} \\
&= \left\| W(x) \int_{a_{\beta n}}^{x} f'(u)\, du \right\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_4 \frac{a_n}{nT(a_n)^{1/2}} \|Wf'\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_5 \frac{a_n}{T(a_{\alpha n})^{1/2} n} \|Wf'\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_6 \frac{a_n}{T(\sigma(t))^{1/2}\, n} \|Wf'\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_7 \frac{a_n}{n} \|Wf'\Phi_t\|_{L_p[x \geqslant a_{\beta n}]} \qquad (6.16)
\end{aligned}$$

by Lemma 4.2, (2.2), and (2.16).

Assume (6.16) holds for 1, 2, ..., $r-1$. Choose $S \in \mathscr{P}_{r-2}$ such that

$$\|W(f'-S)\|_{L_p[|x| \geqslant \sigma(t)]} \leqslant C_6 \left(\frac{a_n}{n}\right)^{r-1} \|f^{(r)}\Phi_t^{r-1} W\|_{L_p(\mathbb{R})}.$$

Set

$$P(x) := f(a_{\beta n}) + \int_{a_{\beta n}}^{x} S(u)\, du.$$

Then we can bound the left hand side of (6.12) by

$$\begin{aligned}
\|W(f-P)\|_{L_p[x \geqslant a_{\beta n}]} &\leqslant C_7 \left\| W(x) \int_{a_{\beta n}}^{x} (f'-S)(u)\, du \right\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_8 \frac{a_n^r}{n^r T(a_n)^{1/2}} \|f^{(r)} W \Phi_t^{r-1}\|_{L_p[x \geqslant a_{\beta n}]} \\
&\leqslant C_9 t^r \|f^{(r)} \Phi_t^r W\|_{L_p(\mathbb{R})} \qquad (6.17)
\end{aligned}$$

and we have our result. ∎

Finally we give:

Proof of Corollary 1.7(b). Write $t=a_u/u$ and let $n=n(t)$ be determined by (1.18).

Firstly

$$\begin{aligned} K_{r,p}(f,W,t^r) &= \inf_{P\in\mathscr{P}_n}\{\|(f-P)W\|_{L_p(\mathbb{R})}+t^r\,\|WP_n^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})}\} \\ &\geqslant \inf_g\{\|(f-g)W\|_{L_p(\mathbb{R})}+t^r\,\|Wg^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})}\} \\ &= K^*_{r,p}(f,W,t^r). \end{aligned} \tag{6.18}$$

Next, we may choose g such that

$$\|(f-g)W\|_{L_p(\mathbb{R})}+t^r\,\|Wg^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})}\leqslant 2K^*_{r,p}(f,W,t^r). \tag{6.19}$$

Also by Lemma 6.1, Theorem 1.3, and Corollary 1.4 we may choose P_n such that

$$\|(P_n-g)W\|_{L_p(\mathbb{R})}\leqslant C_2 w_{r,p}\left(g,W,\frac{a_n}{n}\right) \tag{6.20}$$

and

$$\left(\frac{a_n}{n}\right)^r\|WP_n^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})}\leqslant C_3 w_{r,p}\left(g,W,\frac{a_n}{n}\right). \tag{6.21}$$

Thus by (6.19)–(6.21) we have

$$\begin{aligned} &K_{r,p}(f,W,t^r) \\ &\quad\leqslant \|(f-P_n)W\|_{L_p(\mathbb{R})}+t^r\,\|WP_n^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})} \\ &\quad\leqslant C_4[\|(f-g)W\|_{L_p(\mathbb{R})}+\|(g-P_n)W\|_{L_p(\mathbb{R})}+t^r\,\|WP_n^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})}] \\ &\quad\leqslant C_5\left[\|(f-g)W\|_{L_p(\mathbb{R})}+w_{r,p}\left(g,W,\frac{a_n}{n}\right)\right] \\ &\quad\leqslant C_6[\|(f-g)W\|_{L_p(\mathbb{R})}+w_{r,p}(g,W,t)] \qquad \text{(by (6.2))} \\ &\quad\leqslant C_7[\|(f-g)W\|_{L_p(\mathbb{R})}+t^r\,\|g^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}] \qquad \text{(by Corollary 1.7(a))} \\ &\quad\leqslant C_8 K^*_{r,p}(f,W,t^r). \end{aligned} \tag{6.22}$$

Then (6.18) and (6.22) give the result. ∎

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